Brittle Fracture: Variation of Fracture Toughness with Constraint and Crack Curving under Mode I Conditions

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ABSTRACT—The effect of constraint on brittle fracture of solids under predominantly elastic deformation and mode I loading conditions is studied. Using different cracked specimen geometry, the variation of constraint is achieved in this work. Fracture tests of polymethyl methacrylate were performed using single edge notch, compact tension and double cantilever beam specimens to cover a broad range of constraint. The test data demonstrate that the apparent fracture toughness of the material varies with the specimen geometry or the constraint level. Theory is developed using the critical stress (strain) as the fracture criterion to show that this variation can be interpreted using the critical stress intensity factor $K_C$ and a second parameter $T$ or $A_3$, where $T$ and $A_3$ are the amplitudes of the second and the third term in the Williams series solution, respectively. The implication of this constraint effect to the ASTM fracture toughness value, crack tip opening displacement fracture criterion and energy release rate $G_C$ is discussed. Using the same critical stress (strain) as the fracture criterion, the theory further predicts crack curving or instability under mode I loading conditions. Experimental data are presented and compared with the theory.

KEY WORDS—Brittle fracture, constraint effect, fracture toughness, T-stress

Introduction

Constraint effect in fracture is qualitatively referred to as the dependence of fracture toughness on the specimen thickness, in-plane geometry and loading configuration. For brittle fracture when the crack tip is under large plastic deformation, such as cleavage fracture in ductile metals, it is now recognized that the single fracture parameter $J$ is only valid under the special circumstance when a high degree of constraint at the crack tip exists. In general, the critical $J$ at fracture changes according to the shape and size of the crack configuration. Recent analytical, numerical and experimental studies have attempted to interpret this fracture behavior using both $J$ and a second parameter, $A_2$, $T$ or $O$, where $J$ represents the loading level and the second parameter a constraint level.

For brittle fracture in predominantly elastic deformation mode, current fracture assessment methodology relies on a critical stress intensity factor, $K_{IC}$, or plane strain fracture toughness, which is assumed to be a material property and represents a material’s resistance to fracture under plane strain conditions. Recent studies by Chao and Zhang and Richardson and Goree, however, have shown that the apparent fracture toughness $K_C$ varies with specimen geometry and loading configuration for brittle materials. While this constraint effect is similar to those for ductile materials, the trend for the dependence of the fracture toughness $K_C$ on the constraint is opposite to those for ductile materials. In this paper, $K_C$ is used as the critical stress intensity factor or the apparent fracture toughness for a cracked specimen and $K_{IC}$ is a subset of $K_C$ when the specimen geometry and size requirements in ASTM E399 are met.

The work by Chao and Zhang is based on the following assumptions: (1) the crack tip mechanics fields can be characterized by the Williams series solutions, (b) the fracture event is controlled by a critical stress or critical strain in front of the crack tip and (3) the levels of the critical stress-strain can be characterized by more than one term in the Williams series solutions. The fundamental argument in Ref. 7 is that under certain constraint levels, the stress and strain at a critical distance ahead of the crack tip for brittle materials cannot be characterized by a single parameter $K$ alone. Thus, if two terms in the Williams series are used, the stress (and strain) fields at the critical distance and consequently the fracture event may be characterized by $K$ and another parameter, where $K$ represents the loading level and the second parameter the constraint level. The choice of the second parameter depends on whether the fracture is stress or strain controlled.

The key argument in the work of Chao and Zhang is that the controlling stress-strain for fracture is at a finite distance $r_c$, in contrast to $r \to 0$, from the crack tip and that the stress level at this finite distance $r_c$ may not always be characterized by the first term in the Williams solution, or the $K$-stress. For a given material, if the $K$-dominant zone at a crack tip encompasses the fracture process zone, the actual stress at $r_c$ is close to the $K$-stress. Thus, using a critical stress $\sigma_c$ as the fracture criterion is equivalent to using a critical $K$, such as $K_{IC}$, since these two are related to each other by the first term in the Williams equation. On the other hand, if the $K$-dominant zone is smaller than $r_c$, the stress at $r = r_c$ cannot be completely characterized by the $K$-stress alone. Under these conditions, using $K$ to characterize the fracture event is not sufficient. Furthermore, the size of the $K$-dominant zone depends on the specimen geometry, size, crack length and loading configuration. Thus, the critical stress intensity factor for a given cracked specimen becomes a function of these factors. This behavior can be generally referred to as the constraint effect in brittle fracture.
In this paper, we further examine this constraint effect in the fracture of brittle materials. Governing equations for both stress and strain in the vicinity of a crack tip are revisit. Fracture tests using polymethyl methacrylate (PMMA) plate (acrylic or Plexiglas) were performed to examine the brittle fracture. Compact tension (CT), single edge notch in tension (SENT) and double cantilever beam (DCB) specimens with various crack lengths were used to cover a broad range of constraint levels; that is, the T-stress is negative for SENT, positive for CT and very high for DCB specimens. It is demonstrated that the critical stress intensity factor or apparent fracture toughness of PMMA varies with the specimen geometry or the constraint level. Theory is then developed using the critical stress (strain) as the fracture criterion to show that this variation can be interpreted with the critical stress intensity factor $K_C$ and a second parameter $T$ or $A_3$, where $T$ and $A_3$ are the amplitudes of the second and the third term, respectively, in the Williams series solution. The implication of this constraint effect to the crack tip opening displacement (CTOD) fracture criterion and the energy release rate is then discussed.

In addition to the brittle fracture criterion, crack curving is discussed in the current work because it is an automatic consequence of the discussion. In the fracture mechanics community, crack curving under mode I conditions has been an interesting subject. One of the primary reasons for this interest is that the traditional single parameter fracture criterion cannot explain this phenomenon. Earlier work in this area includes Cotterell\(^9\) and Cotterell and Rice,\(^10\) who postulated that slightly kinked mode I crack may turn back to its original path (stable) or deviate away (unstable) depending on the sign of the coefficient of the second term in the Williams series crack tip solution, that is, stable for $T$-stress $< 0$ and unstable for $T$-stress $> 0$. Cotterell\(^11\) studied the crack curving of DCB specimen geometry and concluded that the crack will propagate in its own direction if $\sigma_y > \sigma_z$ but curve for $\sigma_y < \sigma_z$, where $\sigma_z$ is the opening stress and $\sigma_y$ is the stress parallel to the crack face ahead of the crack tip. The basic assumption used by Cotterell\(^11\) is that the crack propagates in a direction that is normal to the maximum tensile stress, as expected in brittle fracture. Williams and Ewing\(^12\) and Strait and Finnie\(^13\) employed similar arguments but investigated the circumferential or the hoop stress $\sigma_{90}$ at the boundary of the process zone and other terms of the crack tip itself, which better agrees with experimental data. More recent studies include Strait and Finnie,\(^14\) Ramulu and Kobayashi,\(^15\) Thireos and Gdoutos\(^16\) and Kosai et al.\(^17\)\(^,\)\(^18\)

In this paper, we demonstrate that when higher order terms are included in the description of the crack tip stress and strain fields, the maximum hoop stress-strain criterion can predict crack curving under mode I conditions; the reason for this being that the maximum circumferential stress (or strain) occurs at an angle $\theta \neq 0$ degrees, when the contribution from the higher order terms is large. The analysis is presented and the theoretical prediction is then compared with the test results from our DCB specimens, which curved in the tests.

Theoretical Consideration

Fracture of a flawed specimen or structure is a process in which the material separates at the crack tip. Thus, to interpret the fracture behavior, it is natural to examine the stress and strain fields near the crack tip. The mechanics solution for this problem under linear elastic assumptions was obtained by Williams.\(^19\) Using $(r, \theta)$ centered at the crack tip as the coordinate system, the stress and strain can be written as

$$\sigma_{ij} = \sum_{n=1}^{\infty} A_n r^{2n-1} f^{(n)}(\theta)$$

$$\epsilon_{ij} = \sum_{n=1}^{\infty} A_n r^{2n-1} g^{(n)}(\theta),$$

where $\sigma_{ij}$ is the stress tensor, $\epsilon_{ij}$ is the strain tensor and $f^{(n)}(\theta)$ and $g^{(n)}(\theta)$ are the angular functions of the $n$th term for the stress and strain, respectively.

Equations (1) and (2) are asymptotic solutions; that is, as $r$ approaches zero, the contribution of the lower order terms is more important than the contribution of the higher order terms. Note that the first or the lowest order term is singular with respect to $r$. Therefore, one typically assumes that the first term dominates and is the only term that is important because fracture starts at $r = 0$. As such, one has

$$\sigma_{ij} \cong \frac{K_I}{\sqrt{2\pi r}} f^{(1)}(\theta)$$

$$\epsilon_{ij} \cong \frac{K_I}{\sqrt{2\pi r}} g^{(1)}(\theta).$$

In eq (3), $K_I$ is equal to $A_1 \sqrt{2\pi}$ and is the stress intensity factor. It is obvious from eq (3) that the fracture event can be quantified by a single parameter $K_I$ as long as the fracture process is controlled by stress, strain or a combination of stress and strain (e.g., strain energy). The coefficient $K_I$ is proportional to the applied load, and the critical $K_I$ corresponding to the instant of fracture is defined as $K_C$. In this paper, the subscript $C$ is used for the physical quantity at the instant of fracturing (e.g., $T_C$ is the $T$-stress when the fracture event occurs).

Based on eq (3), the material-testing community has designed a testing procedure for the determination of $K_C$ for engineering materials. The testing procedure is designed in a way such that certain restrictions such as specimen geometry and size requirements must be satisfied. The particular $K_C$ determined following the standard testing procedures and satisfying the requirements is labeled $K_{IC}$, or the fracture toughness of the material (see ASTM E399\(^20\)).

Note that the above discussion and the concept of fracture toughness introduced by Irwin\(^21\) and defined by the ASTM testing standards are based purely on the validity of eq (3). In the general case, however, the critical stress or strain near a crack tip that dictates the fracture may not be sufficiently quantified by only the first term in the Williams series, as shown by eq (3). This can be rationalized by considering a fracture process zone at a loaded crack tip. Because the critical distance $r_C$, where the stress-strain dictates the fracture event, is equal to or larger than the size of the process zone, $r_C$ is finite. When $r_C$ is finite, the stress-strain at $r_C$ is better represented by more than simply the first terms from eqs. (1) and (2). If one focuses on the mode I loading case, one needs only examine the stress-strain ahead of the crack tip, that is, along $\theta = 0$ deg. Assuming that two nonvanishing terms in eqs. (1) and (2) are sufficient to characterize the stress-strain at the critical distance, one arrives at
\[
\sigma_{yy}|_{\theta=0} = \frac{A_1}{\sqrt{r}} + 3A_3 \sqrt{r} = \frac{K_I}{\sqrt{2\pi r}} + 3A_3 \sqrt{r} \tag{4}
\]
\[
\varepsilon_{yy}|_{\theta=0} = \frac{1 - v^*}{E^*} \frac{K_I}{\sqrt{2\pi r}} - \frac{v^*}{E^*} T_c \tag{5}
\]
where \(E^* = E \) and \(v^* = v \) for plane stress and \(E^* = \frac{E}{1-v^2} \) and \(v^* = \frac{v}{1-v} \) for plane strain conditions.

In eq (5), \( T = 4A_2 \) and is the so-called \( T \)-stress or the stress parallel to the crack line. Note that this \( T \)-stress could be due to the inherent geometry of a test specimen under mode I loading or to the applied far field loading in the \( x \)-direction. Along \( \theta = 0 \) degrees, the second term in eq (1), or the \( T \)-stresses, vanishes. Thus, three terms or two nonvanishing terms from the Williams expansion are retained in eq (4).

Equation (4) shows that the opening stress is controlled by two mechanics parameters \( K_I \) and \( A_3 \). \( K_I \) can be interpreted as the applied loading level and \( A_3 \) as a function of specimen geometry and loading configuration. Accordingly, eq (5) can be interpreted as two parameters, \( K_I \) and \( T \), that control the opening strain. Note that both \( A_3 \) and \( T \) are proportional to \( K \) and the applied load for a given geometry due to the linearity of the problem.

Adopting the fracture criterion, “The fracture event occurs when the opening stress at \( r_c \) in front of the crack tip reaches a critical value \( \sigma_c \)” or the KRKR model,\(^{22}\) eq (4) becomes
\[
\sigma_c = \frac{K_c}{\sqrt{2\pi r_c}} + 3A_c \sqrt{r_c} \tag{6}
\]
where \( K_c \) and \( A_c \) are the critical \( K_I \) and \( A_3 \), respectively, corresponding to the critical load. Equation (6) shows that the critical stress at \( r_c \) is controlled by two parameters, \( K_c \) and \( A_c \), in contrast to a single parameter presented by eq (3). Because \( A_c \) can be either positive or negative depending on specimen or structural geometry, the magnitude of \( K_c \) varies with specimen geometry in order to achieve the same fracturing stress \( \sigma_c \) for the material to fail. In the special case of \( A_3 = 0 \), the opening stress is controlled by \( K_I \) alone; that is, \( K \) dominance is ensured. Therefore, for this type of specimen, the fracture event is completely controlled by a single parameter \( K \).

Using a similar argument for stress-controlled fracture, one arrives at the following equation for “strain”-controlled fracture:
\[
\varepsilon_c|_{\theta=0} = \frac{1 - v^*}{E^*} \frac{K_I}{\sqrt{2\pi r}} - \frac{v^*}{E^*} T_c \tag{7}
\]
where \( \varepsilon_c \) is the critical opening strain at the critical distance \( r_c \) in front of the crack tip and \( K_c \) and \( T_c \) are the critical stress intensity factor and the critical \( T \)-stress, respectively, at the instant of fracture.

**Experimental Analysis**

To verify the above discussion, specimens of three geometries (i.e., CT, SENT and DCB) with various crack depths were tested. The specimens were chosen in such a way that a wide range of \( T \) or \( A_3 \) could be achieved. To study the brittle fracture behavior, a PMMA plate with thickness \( t = 5.84 \) mm was selected as the material and tested at room temperature. Figure 1 shows the dimensions of the specimen configuration. The crack length (i.e., \( a/W \)) and the specimen designation can be found in the second column of Table 1.

A simple tension test following ASTM E 8-93\(^{23}\) and ASTM E132-92\(^{24}\) standard procedures was performed to determine the elastic material properties for PMMA. Test results show that \( E \) (Young’s modulus) is 2.95 GPa, \( \sigma_{yy} = 44.6 \) MPa and \( v \) (Poisson’s ratio) is 0.34.

All fracture tests were carefully conducted according to ASTM E399-92\(^{25}\) guidelines. The critical stress intensity factor \( K_c \) for each test specimen is determined using a finite element analysis and the fracturing load \( P_Q \) as specified by the ASTM testing standard. The ABAQUS code was used for the finite element analysis. The specimen in the numerical analysis was loaded up to the load \( P_Q \) recorded from the test. The stress intensity factor at this load from numerical analysis is designated as the critical intensity factor or \( K_c \) for the particular specimen. Only the “valid” test data (i.e., satisfying the ASTM requirements in crack front roundness and symmetry) are used for later discussion. The thickness \( t = 5.84 \) mm meets the thickness requirement specified by ASTM E399-92\(^{26}\) for this material; that is, \( t > 2.5(K_{IC}/\sigma_{yy})^2 \), where \( K_{IC} = 1.121 \) MPa\(\cdot\)m\(^{1/2} \) is obtained from specimen CTc3, which is the only CT specimen tested that satisfies all the ASTM E399 requirements.

**Determination of \( T \)-Stress and \( A_3 \) Values**

There are several methods for the determination of \( T \)-stress for a given specimen geometry. In this work, we adopted the method of finding the \( \sigma_{xx} \) along the crack face to determine the \( T \)-stress because of its simplicity. Using eq (1) and setting \( \theta = 180 \) deg, one has
\[
\sigma_{xx} = 4A_2 = T \tag{8}
\]

Figure 2 shows the distribution of \( \sigma_{xx} \) along the crack face from the finite element analysis for a CT specimen with \( a/W = 0.295 \) (i.e., specimen CT1b). The numerical data shown in Fig. 2 indicate that near the crack tip, there exists a nearly constant \( \sigma_{xx} \) region, that is, \( x/a = -0.001 \) to -0.01, and this \( \sigma_{xx} \) value is chosen as the \( T \)-stress.

The \( A_3 \) term in the Williams series expansion was determined by employing eq (4) along with finite element analysis. Equation (4) can be rewritten as
\[
\varepsilon_c|_{\theta=0} \cdot \frac{1 - v^*}{E^*} \frac{K_I}{\sqrt{2\pi r}} = 3A_3 \cdot \frac{v^*}{E^*} T_c \tag{9}
\]
Equation (9) is a simple linear expression:
\[
f(x) = m \cdot x + n \tag{10}
\]
Because \( f(x) \) can be determined by finite element analysis, \( A_3 \) can be obtained from the slope \( m \), that is, \( A_3 = \frac{m}{3v^2} \), through a linear fit. One example is shown in Figure 3 for the CT1b specimen.

The apparent fracture toughness \( K_{IC} \), \( T \)-stress and \( A_3 \) at the critical load \( P_Q \) for each specimen tested are listed in Table 1. The \( T \)-stress is then converted to the dimensionless constraint parameter \( B \) using \( B = \frac{T}{\sqrt{K_{IC}}} \). The values of \( B \) for the test samples at the fracturing load are also included in Table 1.
Interpretation of the Test Results

Our fracture tests show that all the SENT and CT specimens failed without crack curving and all DCB specimens (except one) had curving from the beginning. When crack curving occurs, the application of eqs. (4) through (7) to the problem is not valid because these equations are strictly for stress and strain along θ = 0 deg. As such, only the test data from SENT and CT are used in this section to demonstrate the constraint effect; the data from DCB are used for the interpretation of the crack curving in later sections.

Assuming the fracture event is controlled by an opening stress, eq (6) predicts that the functional form in the $K_C$ and $A_3$ plane will be a straight line. Therefore, using the critical $K_f$ and $A_3$ in Table 1 for both CT and SENT specimens, a straight line is fitted to eq (6) and presented in Fig. 4. This fitted line can be interpreted as the material failure curve for PMMA, and $A_C = 7.57$ MPa and $r_C = 2.32$ mm for this batch of PMMA. Note the slope of the straight line, which indicates that specimens having a lower or more negative $A_3$ fail at higher apparent fracture toughness $K_C$.

Similarly, assuming the fracture event is controlled by an opening strain, eq (7) predicts that the functional form in the $(K_C, T)$ space will be a straight line. Therefore, using the critical $K_f$ and $T$ in Table 1 for both CT and SENT specimens, a straight line is fitted to eq (7) and presented in Figure 5. This fitted line can be interpreted as the material failure curve for PMMA, and $e_C = 2000 \times 10^{-6}$ and $r_C = 0.86$ mm for this
TABLE 1—K_i, T, A_3 AND a/W VALUES FOR SINGLE EDGE NOTCH IN TENSION (SENT), COMPACT TENSION (CT), AND DOUBLE CANTILEVER BEAM (DCB) SPECIMENS

<table>
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<th>a/W (MPa x m^1/2)</th>
<th>K_i (MPa)</th>
<th>T (MPa)</th>
<th>A_3 (MPa x m^-1/2)</th>
<th>B</th>
<th>Did Crack</th>
<th>Curve?</th>
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^aCT3c is the only valid data per ASTM E399.

---

Fig. 4—Polymethyl methacrylate test data plotted in the 
(KC, A_C) plane (SEN = single edge notch in tension, CT = compact tension)

Fitting equation: KC = -1.7431E-02 A_C + 9.1436E-01
Fitting results: r_C = 2.32 mm, a_C = 7.57 MPa

Fig. 5—Polymethyl methacrylate test data plotted in the 
(KC, T_C) plane (SEN = single edge notch in tension, CT = compact tension)

Fitting equation: KC = 7.8161E-02 T_C + 1.0112E+00
Fitting results: r_C = 0.86 mm, a_C = 2000 μs
batch of PMMA. Note the slope of the straight line, which indicates that specimens having lower $T$ values fail at lower apparent fracture toughness $K_C$.

The implication from the above analysis is that the fracture toughness is specimen-geometry dependent. Thus, the fracture toughness value determined following ASTM E399-92 is simply a particular toughness of the material, which satisfies all the ASTM test procedure requirements. When this particular ASTM fracture toughness value is used for specimens of other geometry or structures, it may or may not predict the fracture event accurately or even conservatively.

It is noted that a specimen of deep crack is regarded as high constraint relative to a specimen of shallow crack, as commonly interpreted in fracture mechanics. In general, a deep-cracked specimen has a higher $T$-stress than a shallow-cracked specimen under the same applied $K$. Thus, higher $T$ implies higher constraint. From the data in Figures 4 and 5, it appears that high (low) constraint specimens fail at higher (lower) fracture toughness values. This conclusion is opposite with regard to the behavior of elastic-plastic metals when failure occurs under large-scale plasticity. Detailed discussions on this issue can be found in Chao and Zhang.\(^7\)

**Critical Energy Release Rate $G_C$**

In this section, we assume that the energy release rate $G_C$ for fracture initiation is a constant for a given material and investigate the relation between $G_C$ and the critical stress intensity factor $K_C$. Recall that

$$G = \lim_{\Delta a \to 0} \left( \frac{1}{\Delta a} \int_0^\Delta a \sigma_{\theta 0} \epsilon_{\theta 0} \, dx \right).$$

Using eqs. (4) and (11), eq (15) becomes

$$G = \frac{K_I K'_I}{E^*} + \frac{3\sqrt{2\pi} K'_I A_3}{E^*} \Delta a + \frac{3\sqrt{2\pi} K_I A'_3}{4E^*} \Delta a$$

$$+ \frac{3\pi A_3 A'_3}{4E^*} \Delta a^2,$$

where $K_I$ and $A_3$ are the critical values for crack length equal to $a$ and $K'_I$ and $A'_3$ are the critical values for crack length $a + \Delta a$. Note that when $\Delta a \to 0$, $K'_I \to K_I$ and $A'_3 \to A_3$. So, we have

$$G = \frac{K_I^2}{E^*} + \frac{15\sqrt{2\pi} K_I A_3}{4E^*} \Delta a + \frac{3\pi A_3^2}{4E^*} \Delta a^2.$$  

Ignoring the second higher order term $\Delta a^2$, one has

$$G = \frac{K_I^2}{E^*} + \frac{15\sqrt{2\pi} K_I A_3}{4E^*} \Delta a.$$  

Note that the second term in eq (18) is the additional term, as compared to the widely accepted relation $G = \frac{K_I^2}{E^*}$, due to the retention of the higher-order term $A_3$. Equation (18) indicates that the critical energy release rate $G_C$ is a function of both $K_I$ and $A_3$ or depends on the constraint of the specimen. Equation (18) can be rewritten in the form

$$K_I^2 = \frac{-15\sqrt{2\pi} \Delta a}{4} K_I A_3 + G \cdot E^*.$$  

Figure 6 shows the test data in the $K_I^2$ versus $K_I A_3$ space and the linear fit to the data according to eq (19). The fitted data give $\Delta a = 3.465 \text{ mm}$ and a critical $G$ of $2.486 \times 10^{-4} \text{ MPa\text{-m}}$ corresponding to the case in which $A_3$ vanishes.
The discussion in this section indicates that the critical energy release rate for a given material is a function of specimen constraint level and that the relation \( G = \frac{K_c^2}{K_r} \) is not universally valid for all test specimens and structure members.

**Crack Curving**

The fracture criterion used in the previous sections assumes that the crack propagates in a direction perpendicular to the maximum circumferential stress-strain. If the crack tip stress can be described by the first term in the Williams series solution as eq (3), it can easily be seen that the maximum circumferential (or hoop) stress \( \sigma_{\theta\theta} \) always occurs at \( \theta = 0 \). Thus, the crack would propagate along \( \theta = 0 \) and curving would not occur. However, if higher order terms are included, the maximum hoop stress may or may not occur at \( \theta = 0 \) depending on the magnitude of the higher order terms. Assuming two terms are sufficient to describe the crack tip stress fields, one has

\[
\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}} \left( \frac{3}{4} \cos \theta + \frac{1}{4} \cos \frac{3\theta}{2} \right) + \frac{T}{2} (1 - \cos 2\theta).
\]

This equation can be rearranged into

\[
\frac{\sigma_{\theta\theta}}{K_I} = \frac{3}{4} \cos \theta + \frac{1}{4} \cos \frac{3\theta}{2} + \frac{T}{2} \frac{K_c}{\sqrt{2\pi r}} \left(1 - \cos 2\theta\right).
\]

Equation (21) is plotted in Fig. 7 to show the effect of the \( T \)-stress term on the location of the maximum hoop stress \( \sigma_{\theta\theta} \). As one can see in Figure 7, the maximum \( \sigma_{\theta\theta} \) occurs at \( \theta = 0 \) when \( \frac{T}{\sqrt{2\pi r}} \frac{K_c}{K_I} \) is negative and the peak of the curve shifts to some angle \( \theta > 0 \) when \( \frac{T}{\sqrt{2\pi r}} \frac{K_c}{K_I} \) exceeds some positive number. Mathematically, searching for the maximum of \( \sigma_{\theta\theta} \) yields

\[
\frac{\partial \sigma_{\theta\theta}}{\partial \theta} = \frac{K_I}{\sqrt{2\pi r}} \left[ -\frac{3}{8} \sin \theta - \frac{3}{8} \sin \frac{3\theta}{2} \right] + T \sin 2\theta = 0
\]

(22)

\[
\frac{\partial^2 \sigma_{\theta\theta}}{\partial \theta^2} = \frac{K_I}{\sqrt{2\pi r}} \left[ -\frac{3}{16} \cos \theta - \frac{9}{16} \cos \frac{3\theta}{2} \right] + 2T \cos 2\theta < 0.
\]

(23)

Equation (22) has four solutions:

\[
\theta_c^- = 0, \quad \theta_c^+ = \cos^{-1} \left[ 1 + \sqrt{1 + \frac{1024 \pi^2}{9} \frac{r_c}{K_c} \left( \frac{T_c}{K_c} \right)^2} \right]
\]

\[
\theta_c^- = \pi, \quad \theta_c^+ = 2\pi - \cos^{-1} \left[ 1 - \sqrt{1 + \frac{1024 \pi^2}{9} \frac{r_c}{K_c} \left( \frac{T_c}{K_c} \right)^2} \right].
\]

(24)

It can be shown that the latter two solutions are not physically possible. If the first solution \( \theta_c^- \) holds, eq (23) requires

\[
\frac{T_c}{K_c} < \frac{3}{8} \frac{1}{\sqrt{2\pi r_c}}.
\]

(25)

If the second solution holds, eq (23) requires

\[
\frac{T_c}{K_c} > \frac{3}{8} \frac{1}{\sqrt{2\pi r_c}}.
\]

(26)

Therefore, if \( \frac{T_c}{K_c} < \frac{3}{8} \frac{1}{\sqrt{2\pi r_c}} \), crack curving will not occur and the crack propagation under mode I conditions will be straight and along the \( \theta = 0 \) direction. On the other hand, if \( \frac{T_c}{K_c} > \frac{3}{8} \frac{1}{\sqrt{2\pi r_c}} \), crack curving will occur and the second solution of eq (24) determines the direction of the crack propagation.
Similarly, if three terms from the Williams series solution are used, one has

\[
\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}} \left( \frac{3}{4} \cos \frac{\theta}{2} + \frac{1}{4} \cos \frac{3\theta}{2} \right) + \frac{T}{2} (1 - \cos 2\theta) + 3A_3 \sqrt{r} \left( \frac{5}{4} \cos \frac{\theta}{2} - \frac{1}{4} \cos \frac{5\theta}{2} \right). \tag{27}\]

Using \(\frac{\partial \sigma_{\theta\theta}}{\partial \theta} = 0\) and \(\frac{\partial^2 \sigma_{\theta\theta}}{\partial \theta^2} < 0\), it can be shown that stable crack propagation (or \(\theta^* = 0\)) occurs if \(\frac{I}{K_C} < \frac{13}{8} \frac{A_1}{K_C} \sqrt{r_C} < \frac{1}{8} \frac{1}{\sqrt{2\pi r}}\) and that crack curving occurs if \(\frac{I}{K_I} + \frac{15}{8} \frac{A_1}{K_C} \sqrt{r_C} > \frac{1}{8} \frac{1}{\sqrt{2\pi r}}\). A closed-form solution for the angle \(\theta\) is difficult to derive, and we used MATHCAD to obtain the numerical solutions, as will be shown later.

If the fracture criterion of a critical hoop strain is used, one starts from

\[
\varepsilon_{\theta\theta} = \frac{K_I}{E* \sqrt{2\pi r}} \left[ \left( \frac{3}{4} \cos \frac{\theta}{2} + \frac{1}{4} \cos \frac{3\theta}{2} \right) + \phi \left( \frac{5}{4} \cos \frac{\theta}{2} - \frac{1}{4} \cos \frac{5\theta}{2} \right) \right] \tag{28}
\]

\[
+ \frac{T}{2E*} \left[ (1 - \cos 2\theta) - \phi (1 + \cos 2\theta) \right].
\]

Using \(\frac{\partial \varepsilon_{\theta\theta}}{\partial \theta} = 0\) and \(\frac{\partial^2 \varepsilon_{\theta\theta}}{\partial \theta^2} < 0\), one arrives at

\[
\theta^* = 0 \text{ for } \frac{T}{K_I} < \frac{(3 + \phi)}{8(1 + \phi) \sqrt{2\pi r}}. \tag{29}\]

Also,

\[
\theta_C^* = 2 \cos^{-1} \left[ w - \frac{1}{9} \left( \frac{3ca - b^2}{a^2} \right) - \frac{1}{3} \right] \tag{30}
\]

\[
\text{for } \frac{T}{K_I} > \frac{(3 + \phi)}{8(1 + \phi) \sqrt{2\pi r}},
\]

where the coefficients are determined from MATHCAD as

\[
w = \left[ -\frac{1}{54} -\frac{9cba + 27da^2 + 2b^2}{a^3} + \frac{1}{18} \frac{\sqrt{3}}{a^2} \right] \frac{\sqrt{4c^2a - c^2b^2 - 18cbad + 27d^2a^2 + 4db^2}}{\sqrt{2\pi r}},
\]

\[
a = 4T (1 + \phi), \quad b = -\frac{3}{4} \frac{K_I}{\sqrt{2\pi r}} (1 + \phi), \quad c = -2T (1 + \phi), \quad d = \frac{K_I \phi}{2\sqrt{2\pi r}}.
\]

**Experimental Results for Crack Curving**

The crack-curving criterion based on stress, eq (26), and critical strain, eq (30), is plotted in Fig. 8, as well as the test data. Note that all DCB specimens except one curved from the beginning, and they are predicted very well by the critical stress criterion and slightly off by the critical strain criterion, as shown in Fig. 8. In Fig. 8, the critical distance \(r_C\) determined from Figs. 4 and 5 was used for stress- and strain-controlled fracture, respectively.

The curving angles measured from the broken specimens are presented in Figs. 9 and 10 using the critical stress and critical strain fracture criterion, respectively. All the broken specimens that curved exhibit a smooth curving from the crack tip instead of an abrupt kinking from the crack tip as normally seen in mixed-mode fracture. The angles are measured at the distance \(r_C\) from the crack tip. As shown in Figs. 9 and 10, the predicted angles are higher than all of the measured values. One possible explanation for this discrepancy may be seen from Fig. 7. As shown in Fig. 7, \(\sigma_{\theta\theta}\) in the vicinity of \(\theta = 0\) reaches a plateau as \(\frac{T}{K_I}\) exceeds the critical value \(\frac{T}{K_C} = \frac{1}{8} \frac{1}{\sqrt{2\pi r_C}}\) for crack curving to occur, although the precise location for the maximum is already shifted to an angle \(\theta \neq 0\). Thus, a large scatter is anticipated for the fracturing angles.

**Discussion**

The critical distance of 2.3 mm (from stress-controlled fracture) or 0.86 mm (from strain-controlled fracture) is strictly from crack fitting in the current paper. In a similar discussion by Chao and Zhang,\(^7\) 0.501 mm and 0.031 mm, respectively, were obtained for PMMA of 12.7 mm thick. Recent work by Kosi et al.\(^8\) obtained a critical distance of 1 mm for 2024-T3 aluminum sheets from a biaxially loaded dynamic fracture test. This critical distance may be a function of the specimen material, thickness and/or test temperature. More work is needed to relate this distance to the size of the fracture process zone and understand its physical meaning.

In the current work, we selected specimens having different geometry (i.e., SENT, CT, and DCB) to cover a wide range of constraint (or \(T\)-stress) levels. Alternatively, the use of single-specimen geometry under biaxial, in contrast to uniaxial, loading can also achieve the desired constraint level. Kibler and Roberts\(^2\) used cruciform specimen under biaxial loads. They found that the apparent fracture toughness increased by 50 percent as the biaxial load ratio \(\frac{\sigma_{yy}}{\sigma_{xx}} = 1.5\) for Plexiglas plate and by 25 percent as \(\frac{\sigma_{yy}}{\sigma_{xx}} = 0.25\) for

![Fig. 8—Theoretical prediction compared to test data for crack curving (SEN = single edge notch in tension, CT = compact tension, DCB = double cantilever beam)](image-url)
6061-T4 aluminum alloy, relative to the uniaxial case in which \( \sigma_E(\sigma_E^0) \) was the remotely applied stress parallel (perpendicular) to the crack face. Similar findings were reported by Strait and Finnie\(^{14} \) for 7075-T651 aluminum alloy.

The maximum variation of fracture toughness over the range of \( T \)-stress tested in the current work is only about 30 percent. One may argue that 30 percent is not significant for practical purposes when data scatter is considered. However, the purpose of the current paper is to provide both a fundamental mechanics theory and test data to explain why this variation occurs. The maximum variation in fracture toughness from our tests is limited by the sample geometry (and thus the constraint level) for mode I loading and the material. Provided biaxial testing is available, a larger variation in fracture toughness should be observed because the \( T \)-stress can be varied to a large extent. This conclusion is evidenced by the data presented in Refs. 14 and 26.

It should be pointed out that using a critical stress as the fracture criterion for brittle fracture is rather classical. Griffith\(^{27,28} \) upon realizing that a pure energy balance criterion was not useful for fracture under combined stress conditions, stated that the general condition for rupture would be the attainment of a specific tensile stress at the edge of the crack. Furthermore, in one of his early papers, Irwin\(^{21} \) explicitly pointed out that two parameters \( K \) and \( \sigma_{0K} \) (which is the \( T \)-stress used nowadays) are needed to characterize the crack tip stress fields for various test configurations, loads and crack length. The results presented in the current paper simply follow and confirm Griffith’s and Irwin’s concepts that if the brittle fracture is controlled by stress and strain fields near the crack tip, two critical mechanics parameters, \( K \) and \( T \), are required to characterize the fracture event. It is also interesting to note that Irwin, although facing the same problem that the stress singularity may not exist at the crack tip due to the fracture process zone, opted to increase the effective crack length to compensate for the finite size of the fracture process zone. In doing this, a single parameter \( K \) for controlling the fracture is maintained. Although this approach is apparently adequate for fracture testing under mode I loading, because the variation of the constraint level (or \( T \)-stress) for common testing specimens is not extremely significant, it fails to explain why the biaxial loading changes the fracture toughness values and the direction of fracture.

Note that one of the conclusions from the current experimental and analytical results is that the fracture toughness value and the critical CTOD are not material properties, as many perceived, and depend on the crack and the specimen (or structure) geometry. The fracture toughness \( K_{IC} \), as determined using the procedures outlined by ASTM E399\(^{20} \) is simply the critical stress intensity factor when a test specimen satisfies all the test requirements of the ASTM E399 standard. It is, thus, a subset of the possible critical stress intensity factors that could be encountered from various types of test specimens or flawed structures for a given material. This conclusion, however, by no means de-emphasizes the importance of the ASTM E399 testing standard for fracture toughness value. The ASTM testing standard is a carefully devised testing procedure for the determination of the critical stress intensity factor (defined as the ASTM fracture toughness value \( K_{IC} \) under a particular specimen constraint level (i.e., specific test specimen configuration, size requirement and loading conditions). It therefore provides a reference toughness value for a given material, and this toughness value is invaluable in at least two aspects: (1) when evaluating the fracture toughness value of a given material from different laboratories (e.g., a round-robin program) and (2) when comparing the relative fracture strength of different materials or the same material at different temperatures, since uniform test procedures and a fixed constraint are used by all parties. Because the critical stress intensity factor varies from specimen to specimen (or structure), directly applying the ASTM \( K_{IC} \) (or the critical CTOD) to specimens (or flawed structures) other than those specified by the ASTM for predicting the fracture event is not theoretically sound and may lead to unconservative predictions in practice.

Using \( K_{IC} = 3.106 \sqrt{m} \) from Figure 8, the predicted critical \( a/W \) for curving to occur in the DCB specimens is 0.0578; that is, when \( a/W > 0.0578 \), the crack would curve immediately, and when \( a/W < 0.0578 \), the crack would propagate initially in a straight manner and then start to turn at \( a/W = 0.0578 \), neglecting the dynamic effect. The specimen DCB71(\( a/W = 0.0439 \)) in our test matrix, which did not turn initially in the test, curved eventually starting from approximately \( a/W = 0.06 \) as measured from the broken specimen. This position for the initiation of crack turning is close to the predicted value \( a/W = 0.0578 \).

Conclusion

In this paper, the variation of apparent fracture toughness with specimen configuration or constraint is investigated.
The test data confirm that the critical opening stress or critical opening strain is a valid fracture criterion. Furthermore, two nonvanishing terms in the Williams series solution for characterizing the stress-strain ahead of the crack tip can be used to quantify the constraint behavior. Because the fracture toughness varies with specimen configuration, it follows directly that (1) a common critical CTOD for test specimens and structures for a given material does not exist in general and (2) the relation between the critical energy release rate and the apparent fracture toughness must be modified to include the constraint.

As an extension of the concept of using the critical hoop stress or strain as the fracture criterion, we also demonstrate that crack curving or instability phenomena can be interpreted and predicted. Test data compare well with the curving criterion for predicting when crack curving (or kinking) will occur but is less successful in predicting the quantitative curving angles.

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References