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What is This?
Predictive modeling of nonlinear wave propagation for structural health monitoring with piezoelectric wafer active sensors

Yanfeng Shen and Victor Giurgiutiu

Abstract
This article presents predictive modeling of nonlinear guided wave propagation for structural health monitoring using both finite element method and analytical approach. In our study, the nonlinearity of the guided waves is generated by interaction with a nonlinear breathing crack. Two nonlinear finite element method techniques are used to simulate the breathing crack: (a) element activation/deactivation method and (b) contact analysis. Both techniques are available in ANSYS software package. The solutions obtained by these two finite element method techniques compare quite well. A parametric analytical predictive model is built to simulate guided waves interacting with linear/nonlinear structural damage. This model is coded into MATLAB, and the VWaveFormRevealer graphical user interface is developed to obtain fast predictive waveform solutions for arbitrary combinations of sensor, structural properties, and damage. The predictive model is found capable of describing the nonlinear wave propagation phenomenon. This article finishes with summary and conclusions followed by recommendations for further work.

Keywords
piezoelectric wafer active sensors, nonlinear ultrasonics, Lamb waves, damage detection, structural health monitoring, nondestructive evaluation, breathing crack, higher harmonics

Introduction
Nonlinear ultrasonic technique, which uses distinctive higher harmonics and subharmonics features, proves itself a promising approach to detect incipient changes which are precursors to structural damage (Jhang, 2009; Kruse and Zagrai, 2009). The combined use of guided Lamb waves and nonlinear methods is drawing increasing interest because the nonlinear Lamb waves are endowed with both sensitivity of nonlinear methods and large inspection ranges of guided waves.

To date, most studies on nonlinear ultrasonics have been experimental, demonstrating the capability of nonlinear Lamb waves to detect structural damage (Bermes et al., 2007; Cantrell, 2009; Dutta et al., 2009; Kumar et al., 2009; Nagy, 1998). However, few theoretical predictive studies exist especially for nonlinear Lamb waves. Generation of higher harmonics of Lamb waves has been investigated theoretically (Deng, 1999, 2003), and the existence of antisymmetric or symmetric Lamb waves at nonlinear higher harmonics has been discussed via modal analysis approach and the method of perturbation (Srivastava and di Scalea, 2009).

However, these theoretical studies considered only the situations where nonlinearity are present over the whole domain of wave propagation in the material (mesoscopic nonlinearity); other cases of nonlinear wave propagation, such as wave propagation through localized breathing cracks, are also possible.

When structures are under cyclic fatigue loading, microscopic cracks will begin to form at the structure surface, as shown in Figure 1. They need to be found out before they grow to the critical size and cause catastrophic failures. In our study, we want to know what characteristics the inspection waves will have after interacting with this kind of microscopic cracks, especially when they behave as nonlinear breathing cracks under wave cycles.

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When ultrasonic waves reach a microscopic crack, the crack can be closed and opened under compression and tension, with the compression part of the waves penetrating the crack, while the tension part cannot. The nonlinear phenomenon lies in the fact that the apparent local stiffness of the crack region changes under tension and compression.

The interaction of elastic waves with clapping mechanisms has been studied in the past. Researches on clapping-induced nonlinearities and higher harmonics have been carried out (Biwa et al., 2004, 2006; Richardson, 1979). However, most of these investigations aim at the nonlinearity of elastic bulk waves. Our study focuses on the modeling aspect of contact acoustic nonlinearity (CAN) of Lamb waves, which is a localized nonlinear phenomenon of dispersive guided waves and is different from the previous theoretical studies of nonlinear Lamb waves (Deng, 1999, 2003; Srivastava and di Scalea, 2009).

**Generation of higher harmonics in nonlinear ultrasonics**

A distinctive beneficial feature of nonlinear ultrasonics is the generation of nonlinear higher harmonics, which allows us to diagnose the presence and severity of nonlinear damage in structures. There are various mechanisms behind generation of higher harmonics in ultrasonics, for example, nonlinear mesoscopic (hysteretic) nonlinearity and CAN. The phenomena of higher harmonics generation can be illustrated in a simple way by using a general nonlinear dynamic system (Hagedorn, 1988; Lee and Choi, 2008; Naugolsnykh and Ostrovsky, 1998)

\[ U = A \dot{x} + B \ddot{x}^2 + \cdots \]

where \( U \) is the output of the system, \( A \) is a scale factor, and \( B \) and \( \gamma \) are the second and third nonlinear coefficients. Consider a harmonic input

\[ X(\omega) = \dot{x} \cdot e^{i\omega t} \]  \hspace{1cm} (2)

By substituting equation (2) into equation (1), the output of the nonlinear system takes the form

\[
U = A \ddot{x} + B \ddot{x}^2 + A \gamma (\ddot{x})^3 + \cdots \\
= A \ddot{x} + B \ddot{x}^2 + A \gamma \ddot{x}^3 + \cdots \\
= AX(\omega) + AB \ddot{x} \cdot X(2\omega) + A \gamma \ddot{x}^2 \cdot X(3\omega) + \cdots \]  \hspace{1cm} (3)

Equation (3) shows that the output of the nonlinear system contains higher harmonics \( 2\omega, 3\omega, \ldots \), while the input to the system contains only one frequency component \( \omega \). This distinctive feature allows us to detect material degradation, fatigue, microcracks, or state of clamping surfaces, which introduce nonlinearity to structures.

**Finite element simulation of Lamb waves interacting with nonlinear breathing cracks**

A pitch–catch method may be used to interrogate a plate with a breathing crack which opens and closes under tension and compression. The ultrasonic waves generated by the piezoelectric wafer active sensor (PWAS) propagate into the structure, interact with the breathing crack, acquire nonlinear features, and are picked up by the receiver PWAS. This process is shown in Figure 2.

Two methods are used to model the breathing crack: (a) element activation/deactivation method and (b) contact analysis. The solving scheme and results from both methods are discussed and compared.

**Element activation/deactivation method**

Element activation/deactivation technique could be described as deactivating and reactivating selected elements according to certain criteria. To deactivate elements, the stiffness matrices of the elements are
multiplied by a severe small reduction factor, \( \eta \) (usually 1E-6 or smaller), while mass, damping, loads, and other such effects are set to zero. Thus, upon deactivation, the element stiffness matrix, mass matrix, and associated loads will no longer contribute to the assembled global matrices. It should be noted that, through this approach, the deactivated elements are not removed from the model, but left in place in a dormant state with a greatly diminished participation. Similarly, when elements are activated, they are not added to the model. Instead, the dormant elements are simply reactivated, recovering their original stiffness, mass, damping, element loads, and so on. The assembled global equation will take the following form

Original global equation

\[
\begin{pmatrix}
M_{11} & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & M_{nn}
\end{pmatrix}
\begin{cases}
\ddot{u}_1 \\
\vdots \\
\ddot{u}_n
\end{cases}
+ 
\begin{pmatrix}
C_{11} & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & C_{nn}
\end{pmatrix}
\begin{cases}
\ddot{u}_1 \\
\vdots \\
\ddot{u}_n
\end{cases}
+ 
\begin{pmatrix}
\ddot{u}_1 \\
\vdots \\
\ddot{u}_n
\end{cases}
+ 
\begin{pmatrix}
Q_1 \\
\vdots \\
Q_n
\end{pmatrix} = 0
\]

Deactivated global equation

\[
\begin{pmatrix}
M_{11} & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & M_{nn}
\end{pmatrix}
\begin{cases}
\ddot{u}_1 \\
\vdots \\
\ddot{u}_n
\end{cases}
+ 
\begin{pmatrix}
C_{11} & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & C_{nn}
\end{pmatrix}
\begin{cases}
\ddot{u}_1 \\
\vdots \\
\ddot{u}_n
\end{cases}
+ 
\begin{pmatrix}
\ddot{u}_1 \\
\vdots \\
\ddot{u}_n
\end{cases}
+ 
\begin{pmatrix}
\ddot{u}_1 \\
\vdots \\
\ddot{u}_n
\end{cases}
+ 
\begin{pmatrix}
\ddot{u}_1 \\
\vdots \\
\ddot{u}_n
\end{cases}
+ 
\begin{pmatrix}
\dddot{Q}_1 \\
\vdots \\
\dddot{Q}_n
\end{pmatrix}
\]

where \( M^e, C^e, K^e \), and, \( Q^e \) are the elemental mass matrix, damping matrix, stiffness matrix, and external loads, respectively. The reduction factor \( \eta \) is very small (\( \eta<1 \), typically \( \eta<1E-6 \)). And the symbol \( \Phi \) denotes a zero matrix or vector. Comparing equation (5) with equation (4), it is apparent that the elements, after deactivation, will no longer contribute to the structure because \( \eta K^e=0 \) with \( \eta<1 \). The nonlinear effect is imparted by the periodical change of matrices \( M, C, \) and \( K \).

The solving scheme for this transient dynamic problem using element activation/deactivation method is shown in Figure 3. The crack opening or closing status is judged for each calculation step in the transient analysis; calculation configuration of the current step is based on the results of the previous step.

The crack open/close criterion is developed based on the tension and compression status of the thin layer of nonlinear elements simulating the breathing crack. When these elements are under tension, the crack is considered open. The criterion is shown in the following equation

\[
(U2 - U1 < 0) \cap (\bar{\varepsilon} = \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i < 0)
\]

where \( U1 \) and \( U2 \) are the displacements of the two nodes located on the two edges of the selected element in crack opening direction. \( \bar{\varepsilon} \) is the average strain of the two nodes in crack opening direction. This criterion is developed based on the contact behavior of the breathing crack and through numerical experiments. Details of this criterion can be found in Shen and Giurgiutiu (2012). It should be noted that for mode shapes at high-frequency big plate thickness combination (high \( nd \) value), this criterion needs to be modified by taking into account more nodes across the crack surface to consider the more complicated contact behavior.

Figure 2. Pitch–catch method for the detection of breathing crack; the mode conversion at the crack is illustrated by the two arrows.

Figure 3. Solving scheme of element activation/deactivation method.
Contact analysis with finite element methods

In the physical world, no penetration will happen between contact surfaces; however, in finite element analysis, hypothetical penetration is allowed to ensure equilibrium. The contact parameters are determined by (a) Lagrange multiplier or (b) penalty methods. In this research, the penalty method is adopted. The relationship of penetration and contact tractions is illustrated in Figure 4, where $k$ is the contact stiffness and $\Delta N$ and $\Delta T$ are the normal and tangential penetrations.

The choice of contact stiffness is an important part of contact analysis, because it influences both the accuracy and convergence of the solution, and usually calls for previous experience. When analyzing contact problem, a dilemma will come to us: a small amount of penetration will render more accurate results, so we should chose large contact stiffness; however, this may lead to ill conditioning of the global stiffness matrix and to convergence difficulties. Lower stiffness values can lead to a certain amount of penetration/slip and make the solution easier to converge but give a less accurate solution. Thus, we are searching for a high enough stiffness that the penetration/slip is acceptably small and render a relatively accurate result, but a low enough stiffness that the problem will be well behaved in terms of convergence. ANSYS provides a suggested value of contact stiffness, which will modified by the penalty coefficient to achieve both convergence and accuracy. A common practice is to start from a low contact stiffness which ensures convergence, check if the penetration of the contact surfaces is reasonable, and then increase the penalty coefficient until the surface penetration is reasonably small and solutions between two sequent penalty coefficients do not change. The final contact stiffness used in this study is $7.051 \times 10^{15}$ Pa.

Finite element model for pitch–catch analysis

Figure 5 shows the finite element model of pitch–catch method for detection of nonlinear breathing crack. Two $7\text{mm} \times 7\text{mm} \times 0.2\text{mm}$ PWAS are considered ideally bonded on a 2-mm-thick aluminum plate. The plate is long enough to ensure the received signals are not influenced by boundary reflections. The crack is located at 200 mm from the transmitter, such that the S0 and A0 wave packets have already separated before they arrive at the crack location; hence the S0 and A0 wave packets interact with the breathing crack individually, which allows us to see how the crack interacts with S0 and A0 waves.

The plate is made of aluminum 2024-T3 with Young’s modulus of 72.4 GPa, density of 2700 kg/m$^3$, and Poisson’s ratio of 0.33. The APC-850 material properties are assigned to the PWAS as follows

$$
[C_p] = \begin{bmatrix}
97 & 49 & 49 & 0 & 0 & 0 \\
49 & 84 & 49 & 0 & 0 & 0 \\
49 & 49 & 97 & 0 & 0 & 0 \\
0 & 0 & 0 & 24 & 0 & 0 \\
0 & 0 & 0 & 0 & 22 & 0 \\
0 & 0 & 0 & 0 & 0 & 22
\end{bmatrix} \text{ GPa}
$$

(7)

Figure 5. Nonlinear finite element model of breathing crack.
where \( C_p \) is the stiffness matrix, \( e_p \) is the dielectric matrix, and \( e_p \) is the piezoelectric matrix. The density of the PWAS material is assumed to be \( \rho = 7600 \text{ kg/m}^3 \).

The finite element model is built under the plane strain assumption. PWAS transducers are modeled with coupled field elements (PLANE13) which couple the electrical and mechanical variables (ANSYS 13.0 Multi-Physics). The plate is modeled with four-node structure element PLANE182 with “element birth and death” capability. A 20 vpp 5-count Hanning window modulated sine tone burst signal centered at 100 kHz is applied on the top electrode of the transmitter PWAS. The plate is under free boundary condition.

To solve this problem with good accuracy and high efficiency, a meshing strategy of varying density needs to be performed. The maximum acceptable element size and time step to ensure accuracy are shown in the following equations (Moser et al., 1999)

\[
 l_e = \frac{\lambda_{\text{min}}}{20} \quad (10)
\]

\[
 \Delta t = \frac{1}{20 \ f_{\text{max}}} \quad (11)
\]

For the excitation centered at 100 kHz, we considered the maximum frequency of interest up to 400 kHz, containing up to the third higher harmonic. The dispersion curve is calculated by solving the Rayleigh–Lamb equation and shown in Figure 6(a). The frequency–wavelength relationship is obtained using equation (12) from the dispersion data and plotted in Figure 6(b). The minimum wavelength at 400 kHz appears in A0 mode at 5.478 mm. According to equation (10), the maximum element size should be 0.275 mm. According to equation (11), for 400 kHz, the maximum time step is 0.125 \( \mu \text{s} \)

\[
 \lambda = \frac{c}{f} \quad (12)
\]

Since the mechanical response at crack zone is very complicated, the crack zone is more densely meshed. The region between the breathing crack and the receiver has a mesh size of 0.25 mm (smaller than 0.275) to accurately depict up to the third higher harmonic. A time step of 0.125 \( \mu \text{s} \) is adopted. In the element activation/deactivation method, a very thin layer of nonlinear elements (0.1 mm thick) at the crack zone are selected to be deactivated and reactivated. For the contact analysis, the contact pair is constructed using contact elements (CONTA172 and TARGE169).

The severity of damage is represented by the number of elements selected to be deactivated and reactivated. We define the damage severity as the index where \( r = a/h \) (\( a \) and \( h \) are the crack size and plate thickness, respectively). An index of \( r = 0 \) corresponds to pristine condition, where there is no crack in the plate. In our simulation, we used 20 elements across the thickness at the crack zone. Different damage severities \( r = 0.6, 0.5, 0.4, 0.3, 0.2, 0.1, \) and \( 0 \) are generated by selecting 12, 10, 8, 6, 4, 2, and 0 elements. ANSYS uses an average nodal solution for data postprocessing. Hence, the deactivated elements must be excluded from the average process to avoid result contamination. To highlight the effect of nonlinear wave propagation through a breathing crack, the linear wave propagation through the crack is also investigated.

**Finite element method simulation results and discussions**

The \( r = 0.6 \) case is used as a representative for demonstrating Lamb waves interacting with a breathing crack
and is shown in Figure 7. The same crack behavior could be observed from both the element activation/deactivation method and contact analysis. It is noticed that the tension part of the Lamb waves opens the crack and do not penetrate through it. On the contrary, the compression part of the Lamb waves closes the crack with collision between crack surfaces; hence, the compression part of the Lamb wave can penetrate into the crack.

Figure 8 shows the waveforms of Lamb waves after linear interaction with the crack (Figure 8(a)) and the waveforms of Lamb waves after nonlinear interaction with a breathing crack (Figure 8(b)).

It can be observed that compared with pristine condition, the cracked plate signal has a slight amplitude drop and phase shift in both S0 and A0 packets. Another difference is that a new wave packet appears due to the presence of the crack. This new packet is introduced by mode conversion and contains both S0 packet converted A0 mode and A0 packet converted S0 mode. The linear crack signal is smooth, but the nonlinear breathing crack signal has small zigzags.

The S0, A0, and new wave packets were extracted from the whole time-history using Hanning window and then Fourier transformed. Frequency spectrums of S0, A0, and the new wave packets of $r = 0.6$ case for
linear crack signal and nonlinear breathing crack signal are carried out and plotted in Figures 9 and 10, respectively. For all the wave packets, the pristine signal does not show any higher frequency components. Figure 9 shows, for the linear crack case, all the wave packets show only the fundamental excitation frequency at 100 kHz. It should be noted that there are no higher harmonics for linear interaction between Lamb waves and the crack.

However, the signal from breathing crack plate shows distinctive nonlinear higher harmonics. Figure 10(a) shows nonlinear higher harmonics in the S0 wave packet. Since the excitation frequency is centered at 100 kHz, the 102.8-kHz peak corresponds to the excitation frequency, and the 203.1 and 300.5 kHz correspond to second and third higher harmonics, respectively. It should be noted that the higher harmonics below 400 kHz can be accurately simulated according to the discussions on the mesh size and time step. The frequency components calculated beyond 400 kHz cannot be correctly described and predicted by the finite element mesh employed. For the A0 wave packet (Figure 10(c)), the first peak corresponds to the excitation frequency, and the second higher harmonic could be clearly observed at 198.2 kHz, but the third harmonic is somehow missing. This phenomenon is due to the tuning effect of PWAS and plate structure combination (Giurgiutiu, 2005). The tuning curve shown in Figure 10(d) indicates that at around 300 kHz, where the third harmonic should appear, the A0 mode reaches its rejection point. In other words, for the given PWAS and plate structure, this frequency could not be detected due to the rejection effect at the receiver PWAS. Analysis of the observed “new packet” (Figure 10(b)) also reveals the nonlinear higher harmonics pattern. In this new packet, the feature of nonlinear higher harmonics seems to be more obvious than in the S0 and A0 packets. And the spectral amplitudes of the higher harmonics are closer to that of the excitation.

To diagnose the severity of this nonlinear damage, the results of all the damage severities are compared. The square root of spectral amplitude ratio of second harmonic to excitation frequency is adopted to show the degree of signal nonlinearity, which may serve as a damage index (DI), that is

\[
DI = \sqrt{\frac{A(2f_c)}{A(f_c)}}
\]  \hspace{1cm} (13)

where \(A(f_c)\) and \(A(2f_c)\) denote the spectral amplitude at the excitation frequency and the second higher harmonic. The variation of DI with crack damage intensity is shown for S0 and A0 packets in Figure 11(a) and for the new packet in Figure 11(b). It can be observed that the amplitude ratio DI is relatively small for both S0 and A0 packets, but it is quite big for the new wave packet even at small damage severity. The DI for S0 and A0 has a monotonically increasing relationship with the crack damage intensity. So the DI from the new packet could serve as an early indicator for the presence of a breathing crack, and the DI for the S0 and A0 packets can serve as an indicator of damage severity.

**Comparison of numerical results between two nonlinear finite element methods**

The numerical results from element activation/deactivation method and the contact analysis are compared. The superposed time-domain simulation signals and frequency spectrum from the two finite element methods for \(r = 0.6\) case are shown in Figure 12(a) and (b). It could be observed that the solutions from these two methods agree well with each other. S0 packet has better accuracy; A0 and new packet have slight phase and
amplitude difference. In the frequency spectrum, it could be noticed that at lower frequency range (with two harmonics range) the two methods have good match, but at higher frequency they deviate from each other.

The difference between two solutions are measured and presented by the nondimensional $L_2$ norm

$$
\|u_e - u_c\| = \sqrt{\frac{\sum_{i=1}^{N} (u_e - u_c)^2}{\sum_{i=1}^{N} u_c^2}}
$$

where $u_e$ and $u_c$ are the solutions from element activation/deactivation method and contact analysis and $N$ is the number of solution points in the time-domain signal. The $L_2$ norm values for $r = 0.1, 0.2, 0.3, 0.4, 0.5, \text{and } 0.6$ cases are plotted in Figure 13. It could be observed that for all the damage severity cases, both methods match stably well with each other.

**Analytical modeling of Lamb waves interacting with nonlinear structural damage**

Figure 14 shows the pitch–catch active sensing method for damage detection: the T-PWAS transducer generates ultrasonic-guided waves which propagate into the structure, interact with structural damage at $x = x_d$, carry the damage information with them, and are picked up by the R-PWAS transducer at $x = x_r$.

To model the damage effect on Lamb wave propagation, we consider the damage as a new wave source at $x = x_d$ and we add mode conversion and nonlinear sources at the damage location through damage interaction coefficients. The predictive analytical model for Lamb wave interaction with damage is constructed in frequency-domain in the following steps:

**Step 1.** Perform Fourier transform of the time-domain excitation signal $V_T(t)$ to obtain the frequency-domain excitation spectrum, $\tilde{V}_T(\omega)$. 

![Figure 10. Frequency spectrum of the Lamb wave signals after nonlinear interaction with a crack: (a) S0 mode, (b) new packet, (c) A0 mode, and (d) tuning curves for A0 and S0 modes explaining the missing A0 peak in (c). Note the presence of distinctive nonlinear higher harmonics.](image-url)
Step 2. Calculate the frequency-domain structural transfer function $G(x_r, \omega)$ from T-PWAS to R-PWAS. The structure transfer function $G(x_r, \omega)$ is given by equation (99) of Giurgiutiu (2007: 327), which gives the in-plane wave strain at the plate surface as

$$
e_x(x,t) = -i \frac{\alpha \tau_0}{\mu} \left\{ \sum_{\xi} \left( \sin \xi x a \right) \frac{N_S(\xi^4)}{D_S(\xi^4)} e^{-i(\xi x - \omega t)} + \sum_{\xi'} \left( \sin \xi' x a \right) \frac{N_A(\xi'^4)}{D_A(\xi'^4)} e^{-i(\xi' x - \omega t)} \right\}$$

where $\xi$ is the frequency-dependent wavenumber of each Lamb wave mode and the superscripts $S$ and $A$ refer to symmetric and antisymmetric Lamb wave modes. If only the two fundamental modes, S0 and A0, are present, then $G(x_r, \omega)$ can be written as

$$G(x_r, \omega) = S(\omega) e^{-i\xi x_r} + A(\omega) e^{-i\xi' x_r}$$

$$S(\omega) = \kappa_{\text{PWAS}} \sin \xi^2 a \frac{N_S(\xi^4)}{D_S(\xi^4)}$$

$$A(\omega) = \kappa_{\text{PWAS}} \sin \xi'^2 a \frac{N_A(\xi'^4)}{D_A(\xi'^4)}$$

Figure 11. Damage severity index: (a) S0 and A0 packets and (b) new packet.

Figure 12. Comparison between signals from element activation/deactivation method and contact analysis (a) time-domain signal and (b) frequency spectrum.

PWAS: piezoelectric wafer active sensor.
\[
N_i(\xi) = \xi \beta (\xi^2 + \beta^2) \cos \theta t \cos \phi d;
\]
\[
D_i(\xi) = (\xi^2 - \beta^2)^2 \cos \phi d \sin \beta d + 4\xi^2 \alpha \beta \sin \phi d \cos \beta d \sin \theta t d;
\]
\[
N_d(\xi) = \xi \beta (\xi^2 + \beta^2) \sin \phi d \sin \beta d;
\]
\[
D_d(\xi) = (\xi^2 - \beta^2)^2 \sin \phi d \cos \beta d + 4\xi^2 \alpha \beta \cos \phi d \sin \beta d \sin \theta t d;
\]
\[
\alpha^2 = \frac{\omega^2}{c_p^2} - \xi^2; \quad \beta^2 = \frac{\omega^2}{c_s^2} - \xi^2; \quad c_p = \sqrt{\frac{\lambda + 2\mu}{\rho}}; \quad c_s = \sqrt{\frac{\mu}{\rho}} \quad \kappa_{\text{PWAS}} = -\frac{i \sigma \tau_0}{\mu};
\]
(17)

where \(\kappa_{\text{PWAS}}\) is the complex transduction coefficient that converts applied voltage into guided wave strain at the \(T\)-PWAS, \(\phi\) a half length of PWAS size, and \(\phi\) plate half thickness. The modal participation functions \(S(\omega)\) and \(A(\omega)\) determine the amplitudes of the S0 and A0 wave modes. The terms \(\sin(\xi^2 \alpha)\) and \(\sin(\xi^2 \alpha)\) control the tuning between the PWAS transducer and the Lamb waves. \(\lambda\) and \(\mu\) are Lamé’s constants of the structural material; \(\rho\) is the material density. The wavenumber \(\xi\) of a specific mode for certain frequency \(\omega\) is calculated from Rayleigh–Lamb equation

\[
\tan \beta d = \frac{\tan \phi d}{\tan \alpha d} = \left[ \frac{-4\alpha \beta \xi^2}{(\xi^2 - \beta^2)^2} \right]^{\pm 1}
\]
(18)

where \(+1\) exponent corresponds to symmetric Lamb wave modes and \(-1\) exponent corresponds to antisymmetric Lamb wave modes.

**Step 3.** Multiply the structural transfer function by frequency-domain excitation signal to obtain the frequency-domain signal at the damage location, that is, \(\tilde{V}_D(\omega) = G(x_d, \omega) \cdot \tilde{V}_T(\omega)\). Hence, the signal at the damage location is

\[
\tilde{V}_D(x_d, \omega) = S(\omega) \tilde{V}_T(\omega)e^{-i \xi^2 x_d} + A(\omega) \tilde{V}_T(\omega)e^{-i \xi^2 x_d}
\]
(19)

This signal could be decomposed into symmetric and antisymmetric components

\[
\tilde{V}_D^S(x_d, \omega) = S(\omega) \tilde{V}_T(\omega)e^{-i \xi^2 x_d}
\]
(20)

\[
\tilde{V}_D^A(x_d, \omega) = A(\omega) \tilde{V}_T(\omega)e^{-i \xi^2 x_d}
\]
(21)

**Step 4.** The wave signal at the damage location takes the damage information by considering transmission, reflection, mode conversion, and higher harmonics. Each of these addition phenomena is modeled as a new wave source at the damage location using damage interaction coefficients (Figure 15). We distinguish two damage interaction types: (a) linear and (b) nonlinear, as discussed next.

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**Figure 13.** Difference between two solutions for various damage severities.

**Figure 14.** A pitch–catch configuration between a transmitter PWAS and a receiver PWAS. PWAS: piezoelectric wafer active sensor.
**Linear damage interaction.** Wave transmission, reflection, and mode conversion are realized by using complex-amplitude damage interaction coefficients. Our notations are as follows: we use three letters to describe the interaction phenomena, with the first letter denoting the incident wave type, the second letter standing for resulting wave type, and the third letter meaning propagation direction (transmission/reflection). For instance, SST (symmetric-symmetric-transmission) means incident symmetric waves transmitted as symmetric waves, while SAT (symmetric-antisymmetric-transmission) means incident symmetric waves transmitted and mode converted to antisymmetric waves. Thus, the complex-amplitude damage interaction coefficient \( C_{i,v}^{M} \cdot e^{i\phi_{v}^{M}} \) denotes the transmitted symmetric mode generated by incident symmetric mode with magnitude \( C_{i}^{M} \) and phase \( \phi_{v}^{M} \).

**Step 5.** The guided waves from the new wave sources created at the damage location propagate through the rest of the structure and arrive at the R-PWAS. The received wave signal is calculated in frequency-domain as

\[
\tilde{V}_{R}(x_{d}, x_{r}, \omega) = \sum_{M=1}^{M} \left[ C_{SST}^{M} e^{-i\phi_{SST}^{M}} \cdot \tilde{V}_{MD}^{S}(x_{d}, \omega) + C_{AAT}^{M} e^{-i\phi_{AAT}^{M}} \cdot \tilde{V}_{MD}^{A}(x_{d}, \omega) \right] e^{-i\omega(x_{r}-x_{d})}
\]

where \( M \) is the number of higher harmonics considered. For linear interaction with damage, \( M \) equals to one.

**Step 6.** Perform inverse Fourier transform to obtain the time-domain receiver sensing signal

\[
V_{R}(x_{d}, x_{r}, t) = \text{IFT} \{ \tilde{V}_{R}(x_{d}, x_{r}, \omega) \}
\]

The analytical procedure is coded in MATLAB and resulted in the graphical user interface (GUI) called WFR as shown in Figure 16(a). Full details of this GUI and MATLAB code are available in Shen and Giurgiutiu (2012). The linear interaction between guided waves and damage is described by the transmission, reflection, and mode conversion parameters as shown in Figure 16(b). For example, SST represents the magnitude of transmitted S0 mode generated by an incoming S0 mode; whereas SAT and phi-SAT represent the magnitude and phase of the transmitted A0 mode resulting from the mode conversion of an incoming S0 mode.

For the purpose of this study, we have also introduced nonlinear parameters representing the result of the nonlinear interaction between the incoming guided
waves and the nonlinear damage. The nonlinear parameters are represented by the magnitude and phase of the second and third harmonic waves (transmitted, reflected, mode converted).

The values of these damage-interaction coefficients are not calculated by the WFR code. These coefficients may be determined experimentally or calculated through other methods: analytical, finite element method (FEM), boundary element method (BEM), etc. In this study, we used a trial-and-error approach to tune the WFR coefficients to the data simulated by the FEM analysis (similar tuning could be done with experimental data, and this approach may be tried in a future study). The tuning procedure is taken out via comparing the analytical solution with FEM results and adjusting the damage interaction coefficients in WFR until both results match with each other. The beneficial aspect of this analytical model is that one would not need to run the FEM model for the whole geometric domain. A local FEM mesh can provide the damage interaction coefficients. A local–global method then could be applied to find the predictive sensing signal (Gresil and Giurgiuțiu, 2013a, 2013b). This will greatly enhance the computational efficiency of the target problem.

WFR allows users to conduct fast parametric studies. It may take several hours for commercial finite element software to obtain an acceptable-accuracy solution for high-frequency, long-distance propagating waves; but it takes only several seconds to obtain the same predictive solution with WFR. Besides, the WFR allows the user to play with all the parameters: PWAS size, plate material properties, sensor/damage locations, and damage type (linear/nonlinear damage with various severities).

**Predictive solution of parametric analytical model for nonlinear wave propagation**

The parametric analytical model is used to predict the nonlinear waveform of finite element simulations. The transmission, mode conversion coefficients, and phase information are obtained from the finite element results. In our analytical model, only the first three harmonics are considered (totally 12 variables need to be defined). The coefficients are shown in Table 1 and input into the WFR.

Figure 17 shows the comparison between finite element simulation and analytical solution from WFR. It is noticed that once the parameters for the analytical solution are given, the finite element simulation result and the analytical solution agree well with each other. The time-domain waveforms share the same nonlinear characteristics of noticeable zigzags in the new packet, and the frequency spectrums match well with each other as well. Since we only consider up to the third higher harmonic in this parametric analytical case study, the frequency-domain of analytical solution shows only the first three peaks, while the finite element
solution have even higher harmonics. But the solution up to the third higher harmonics is accurate enough to render a decent waveform in time-domain. Given the damage-interaction parameters (DIPs), this predictive model can well describe high-frequency, long-distance, linear/nonlinear wave propagation.

**Summary, conclusions, and future work**

**Summary**

In this study, we presented predictive modeling of nonlinear guided wave propagation for structural health monitoring using both FEM and analytical approach. The nonlinearity of the guided waves was generated by interaction with a nonlinear breathing crack. Two nonlinear FEM techniques were used to simulate the breathing crack: (a) element activation/deactivation method and (b) contact analysis. The solutions obtained by these two FEM techniques compared quite well. A linear FEM analysis of this situation was also performed. A parametric analytical predictive model was built to simulate guided waves interaction with linear/nonlinear structural damage. This model was coded into MATLAB, and the WFR GUI was developed to obtain fast predictive waveform solutions for arbitrary combinations of sensor, structural properties, and damage.

**Table 1.** Magnitude and phase parameters to input into analytical solution.

<table>
<thead>
<tr>
<th>Magnitude coefficient</th>
<th>$C_{1}^{SST}$</th>
<th>$C_{1}^{SAT}$</th>
<th>$C_{1}^{AAT}$</th>
<th>$C_{1}^{AST}$</th>
<th>$C_{2}^{SST}$</th>
<th>$C_{2}^{SAT}$</th>
<th>$C_{2}^{AAT}$</th>
<th>$C_{2}^{AST}$</th>
<th>$C_{3}^{SST}$</th>
<th>$C_{3}^{SAT}$</th>
<th>$C_{3}^{AAT}$</th>
<th>$C_{3}^{AST}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value (normalized)</td>
<td>0.900</td>
<td>0.420</td>
<td>0.820</td>
<td>0.100</td>
<td>0.082</td>
<td>0.100</td>
<td>0.050</td>
<td>0.110</td>
<td>0.032</td>
<td>0.038</td>
<td>0.005</td>
<td>0.025</td>
</tr>
<tr>
<td>Phase coefficient</td>
<td>$\phi_{1}^{SST}$</td>
<td>$\phi_{1}^{SAT}$</td>
<td>$\phi_{1}^{AAT}$</td>
<td>$\phi_{1}^{AST}$</td>
<td>$\phi_{2}^{SST}$</td>
<td>$\phi_{2}^{SAT}$</td>
<td>$\phi_{2}^{AAT}$</td>
<td>$\phi_{2}^{AST}$</td>
<td>$\phi_{3}^{SST}$</td>
<td>$\phi_{3}^{SAT}$</td>
<td>$\phi_{3}^{AAT}$</td>
<td>$\phi_{3}^{AST}$</td>
</tr>
<tr>
<td>Value (°)</td>
<td>0</td>
<td>100</td>
<td>-35</td>
<td>90</td>
<td>0</td>
<td>0</td>
<td>120</td>
<td>90</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Figure 17.** Comparison between FEM and analytical simulation (WFR).

FEM: finite element simulation; WFR: WaveFormRevealer.
Conclusions

It was found that the two FEM methods considered in this study can simulate equally well the nonlinear behavior of the breathing crack. It was found that the nonlinear interaction between guided waves and the breathing crack generates higher harmonics which were not found in the linear FEM simulation. A DI was proposed based on the amplitude ratio of the signal spectral harmonics to relate the signal nonlinearity with damage severity. This DI was applied to the S0 and A0 wave packets as well as to a new packet resulting from the interaction between the guided waves and the damage. It was found that the DI of the new packet is more sensitive to the presence of the crack, while the DIs of the S0 and A0 packets can provide monitoring information on the damage severity. It was found that the analytical predictive model WFR can predict the nonlinear effect in the signal using DIPs which were obtained by “trial and error.” It was also found that computational time savings of several orders of magnitude are obtained by using the analytical model WFR instead of FEM methods.

Future work

The behavior of breathing crack under different interrogating wave amplitude should be studied, as well as the transition requirement from initially opened or closed crack into breathing crack. Experiments should be performed to verify these theoretical predictions. Rational methods of determining DIP values need to be found (not trial and error). Work should be carried out to extend the analysis to two-dimensional (2D) wave propagation (three-dimensional (3D) FEM and 2D WFR).

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References
