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What is This?
Exact analytical modeling of power and energy for multimode lamb waves excited by piezoelectric wafer active sensors

Ayman M Kamal, Bin Lin and Victor Giurgiutiu

Abstract
This article presents an analytical model for power and energy transfer between excited piezoelectric wafer active sensors and host structure. This model is based on exact multimodal Lamb waves, normal mode expansion technique, and orthogonality of Lamb waves. Modal participation factors are presented to show the contribution of every mode to the total energy transfer. The model assumptions include the following: (1) straight-crested multimodal ultrasonic guided wave propagation, (2) propagating waves only, (3) ideal bonding (pin-force) connection between piezoelectric wafer active sensors and structure, and (4) ideal excitation source at the transmitter piezoelectric wafer active sensors. Constrained piezoelectric wafer active sensor admittance is reviewed. Electrical active power, mechanical converted power, and Lamb wave kinetic and potential energies are derived in closed-form formulae. Numerical simulations are performed for the case of symmetric and antisymmetric excitation of thin aluminum structure. The simulation results are compared with axial and flexural approximation for the case of low-frequency Lamb waves. In addition, a thick steel structure example is considered to illustrate the case of multimodal guided waves. A parametric study for different excitation frequencies and different transducer sizes is performed to show the best match of frequency and piezoelectric wafer active sensor size to achieve maximum energy transfer into the excited structure.

Keywords
Structural health monitoring, piezoelectric wafer active sensors, multimode guided Lamb waves, wave power, wave energy, nondestructive evaluation, ultrasonic power, impedance, admittance, lead zirconate titanate, normal mode expansion

Introduction
Structural health monitoring (SHM) is crucial for monitoring structure performance and detecting the initiation of flaws and damages in order to predict structural life. SHM uses permanently attached sensors to the structure. Using piezoelectric wafer active sensors (PWAS) has the following advantages: (1) low cost and (2) they serve as passive sensors, that is, without interacting with the structure, and/or (3) active sensors, where they interact with the structure to detect the presence and intensity of damage (Giurgiutiu, 2010). Guided waves are commonly used in nondestructive evaluation (NDE) techniques, such as pitch–catch, pulse–echo, and phased array (Figure 1).

Ultrasonic Lamb waves are used for finding damages and flaws in plates, pipes, rails, thin-walled structures, multilayered structures, and composite materials. The advantage of Lamb waves over other common ultrasonic techniques is that they travel at large distance along the structure. Lamb waves can be “tuned” to excite certain modes; some modes are more sensitive for certain types of defects.

Chinthalapudi and Hassan (2005) showed that energy loss of guided waves may be due to multiple reasons, among them is existing flaws in the structure. Impedance mismatch is considered as “energy-stealing” agent, and flaws such as delamination, split, and cavities cause this. In practice, the sensitivity to most simple defects such as notches and cracks is adequate and of similar magnitude due to the fairly uniform...
distribution of energy through the thickness of the plate; the sensitivity considerations become much more important in anisotropic materials. Wilcox et al. (2001) showed an example of delamination detection in composites where certain modes were found to be blind to delamination at certain depths. Other studies (Alleyne and Cawley, 1992; Koh et al., 2002) gave more insights on how different defects interact with Lamb waves and how severity of impact damages can be predicted from the transmitted power. Generally, failure theories based on energy methods are more robust in predicting failure. Hence, it is very important to model Lamb wave power and energy transduction between PWAS and host structures. Another application that attracts more interests recently is energy harvesting applications. The need of optimizing energy transfer (Kural et al., 2011; Park et al., 2007) requires accurate models for Lamb wave energy, rather than simplified axial and flexural approximation valid only at low frequencies.

Excitation at frequencies beyond the cutoff frequency of $A_1$ and $S_1$ modes will generate multimodal Lamb waves. This phenomenon appears also for relatively thick structures. In these cases, every mode shares parts of the supplied power and energy. Our analytical model is developed based on “normal modes.” Normal modes represent the possible vibration characteristics of the structure and are independent on loading scheme (Rose, 1999). The method of normal mode expansion (NME) is described in this study. It is worth mentioning that there are other methods that can be used to solve forced loading of a structure, for example, the integral transform techniques (ITT). Some of the most popular transforms are Laplace, Fourier, Hankel, and Mellin. Various integral transforms are used to transform a given function into another; this transformation is done via integration (over some domain) of the original function multiplied by a known kernel function. This is followed by either solving for the exact solution, for example, with residue theorem, or by numerically evaluating the integral in the case of complicated problems.

The solution of Lamb wave propagation in a plate that is excited with surface PWAS was obtained by Giurgiutiu (2008) with integral transform technique of the exact solution. NME method determines the expanded amplitudes. NME can be used for isotropic or generally anisotropic layers. The difference between isotropic and anisotropic cases is in evaluating the quantities appearing in the solution. Therefore, the NME method can be considered more general because the physical nature of the excitation process is clear and independent on the material. ITT method does have extensive algebra, and Viktorov (1967) discussed the method of ITT in detail and how the solution is different between isotropic and generally anisotropic layers.

Figure 1. The various ways in which PWAS are used for structural sensing includes (a) propagating Lamb waves, (b) standing Lamb waves and (c) phased arrays. The propagating waves methods include: pitch-catch; pulse-echo; thickness mode; and passive detection of impacts and acoustic emission (Giurgiu tiu, 2008).

PWAS: piezoelectric wafer active sensors; AE: acoustic emission.
Lamb wave NME

NME method is used to (1) find directly the amplitudes of given mode in terms of loading parameters and (2) evaluate the contribution factor of every mode to the total wave power and energy. Normal modes of the guided waves in the structure serve like the eigenfunctions. The method assumes that the desired solution can be written in the form of series of known functions, each with unknown amplitudes. Then, those amplitudes are to be determined either numerically or by finding a general expression that is valid for all modes.

Normal modes (eigenfunctions) of the analyzed structure are assumed “complete,” meaning that any function can be represented exactly in terms of a finite or infinite number of functions in the set of “normal modes.” Second condition for NME method is the orthogonality of the base functions (Rose, 1999). NME method is used to (1) find directly the amplitudes of point P1 due to force \( \mathbf{F}_1 \) and \( \mathbf{u}_{12} \) is the displacement induced by force \( \mathbf{F}_2 \) at the upper and lower surfaces \((y = \pm d)\), where \( d \) is the plate half thickness.

\[ \frac{\tan \alpha d}{\tan \beta d} = -\left[ \frac{\left( \xi_s^2 - \beta_s^2 \right)^2}{4\xi_s^2 \alpha \beta} \right]^{1/2} \]

For free wave motion, the homogeneous solution is derived by applying the stress-free boundary conditions at the upper and lower surfaces \((y = \pm d)\), where \( d \) is the plate half thickness.

\[ R_S = \left( \frac{\xi_s^2 - \beta_s^2}{2\xi_s \beta_s \cos \alpha} \right), \quad R_A = \left( \frac{\xi_A^2 - \beta_A^2}{2\xi_A \beta_A \sin \alpha} \right) \]

Reciprocity relation for Lamb waves

Reciprocity relation is more or less an extension of Newton’s third law of motion, where action and reaction are equivalent. Assume that \( \mathbf{u}_{12} \) is the displacement of point \( P_1 \) due to force \( \mathbf{F}_2 \) and \( \mathbf{u}_{21} \) is the displacement of point \( P_2 \) due to force \( \mathbf{F}_1 \). In its most elementary form, the mechanics reciprocity principle states that (Santoni, 2010) the work done at point \( P_1 \) by force \( \mathbf{F}_1 \) upon the displacement induced by force \( \mathbf{F}_2 \) is the same as the work done at point \( P_2 \) by force \( \mathbf{F}_2 \) upon the displacement induced by force \( \mathbf{F}_1 \), that is

\[ \mathbf{F}_1 \cdot \mathbf{u}_{12} = \mathbf{F}_2 \cdot \mathbf{u}_{21} \]

For Lamb waves, one has real reciprocity and complex reciprocity; we focus on complex reciprocity following Auld (1990).

Considering a generic body \( \Omega \) and two sources \( \mathbf{F}_1 \) and \( \mathbf{F}_2 \) applied at points \( P_1 \) and \( P_2 \) (Figure 3), the two
force sources produce two wave fields with velocity and stress \( \mathbf{v}_1, \mathbf{T}_1 \) and \( \mathbf{v}_2, \mathbf{T}_2 \). Using equation of motion and applying the two different sources (1) and (2) and adding the two field equations together, we can prove the complex reciprocity form that relates the velocity responses, tractions, and applied sources for harmonic excitation, that is

\[
\nabla \left( \tilde{\mathbf{v}}_2 \cdot \mathbf{v}_1 + \mathbf{v}_1 \cdot \tilde{\mathbf{T}}_2 \right) = - \left( \tilde{\mathbf{v}}_2 \cdot \mathbf{F}_1 + \mathbf{v}_1 \cdot \tilde{\mathbf{F}}_2 \right) \tag{7}
\]

For Lamb waves, a similar relation has been derived in Santoni (2010), with the assumption of time harmonic solution. One important assumption considered throughout the analysis is that Lamb wave fields are z-invariant. Hence, the only surviving stresses are normal stresses \( T_{xx} \) and \( T_{yy} \) and shear stress \( T_{xy} \), \( v_x \) and \( v_y \) are the velocity fields, and superscripts 1 and 2 indicate fields due to sources 1 and 2.

The complex reciprocity relation for Lamb waves takes the form

\[
\frac{\partial}{\partial x} \left( \tilde{v}_x^2 T_{xx}^1 + \tilde{v}_y^2 T_{xy}^1 + v_x^1 T_{xx}^2 + v_y^1 T_{xy}^2 \right) + \frac{\partial}{\partial y} \left( \tilde{v}_x^2 T_{xy}^1 + \tilde{v}_y^2 T_{yy}^1 + v_x^1 T_{xy}^2 + v_y^1 T_{yy}^2 \right) = - \tilde{v}_x^2 F_x - v_x^1 F_x^2 - \tilde{v}_y^2 F_y - v_y^1 F_y^2 \tag{8}
\]

This reciprocity relation is the basic formula used to derive orthogonality condition; in addition, the source influence (PWAS excitation) determines modal contribution factors for each mode.

**Orthogonality of Lamb waves**

The definition of orthogonal functions \( U \) over given domain \([a, b]\) can be defined as

\[
\int_a^b U_m U_n dy = 0 \quad \text{for} \ m \neq n \tag{9}
\]

Recalling the complex reciprocity relation of equation (7) with the source forces \( F_x \) and \( F_y \) = 0 and assuming 1 and 2 are two solutions for time-harmonic-propagating Lamb waves, we get

\[
\mathbf{v}_1(x, y, z, t) = \left( \tilde{v}_x^p(y) \mathbf{x} + \tilde{v}_y^p(y) \mathbf{y} \right) e^{-i \xi_y e^{i \omega t}} \tag{10}
\]

\[
\mathbf{v}_2(x, y, z, t) = \left( \tilde{v}_x^m(y) \mathbf{x} + \tilde{v}_y^m(y) \mathbf{y} \right) e^{i \xi_y e^{-i \omega t}}
\]

\[
\mathbf{T}_1(x, y, z, t) = \begin{bmatrix} T_{xx}^n(y) & T_{xy}^n(y) & 0 \\ T_{xy}^n(y) & T_{yy}^n(y) & 0 \\ 0 & 0 & T_{zz}^n(y) \end{bmatrix} e^{-i \xi_y e^{i \omega t}} \tag{11}
\]

\[
\mathbf{T}_2(x, y, z, t) = \begin{bmatrix} \tilde{T}_{xx}^m(y) & \tilde{T}_{xy}^m(y) & 0 \\ \tilde{T}_{xy}^m(y) & \tilde{T}_{yy}^m(y) & 0 \\ 0 & 0 & \tilde{T}_{zz}^m(y) \end{bmatrix} e^{i \xi_y e^{-i \omega t}}
\]

Substituting equations (10) and (11) in the reciprocity equation (7) with \( F_x \) and \( F_y \) = 0 and integrating over plate thickness, we get

\[
- i \left( \xi_n - \tilde{\xi}_m \right) \int_a^b \left( \tilde{v}_x^p(y) \tilde{T}_{xx}^m(y) + \tilde{v}_y^p(y) T_{yy}^m(y) - v_x^m(y) \tilde{T}_{xx}^p(y) \right) dy = i \left( \xi_n - \tilde{\xi}_m \right) 4 P_{mn} = 0
\]

Alternatively, in short form

\[
\int_a^b \left( \tilde{v}_x^p(y) \tilde{T}_{xx}^m(y) + \tilde{v}_y^p(y) T_{yy}^m(y) - v_x^m(y) \tilde{T}_{xx}^p(y) \right) dy = 0 \tag{13}
\]

where

\[
P_{mn} = - \frac{1}{4} \int_a^b \left( \tilde{v}_x^p(y) \tilde{T}_{xx}^m(y) + \tilde{v}_y^p(y) T_{yy}^m(y) - v_x^m(y) \tilde{T}_{xx}^p(y) \right) dy \tag{15}
\]

Recall our assumption of considering only propagating waves (evanescent waves, which die out away from the source, are ignored); consequently, \( \xi_m \) and \( \xi_n \) are real and \( \tilde{\xi}_m = \tilde{\xi}_n \). Since \( \text{Re}(\mathbf{a} \cdot \mathbf{b}) = \text{Re}(\mathbf{a} \cdot \mathbf{b}) \), the orthogonality condition can be further simplified to

\[
P_{mn} = \begin{cases} 0 & \text{if } m \neq n \\ \text{Re} \left[ \frac{1}{4} \int_a^b \left( \tilde{v}_x^p(y) T_{xx}^m(y) + \tilde{v}_y^p(y) T_{yy}^m(y) \right) dy \right] & \text{if } m = n \end{cases} \tag{16}
\]

Figure 3. Reciprocity relation.
\( P_{mn} \) is a measure of average power flow through the plate and is used to determine Lamb wave amplitudes through normalization.

**Normalization of wave amplitudes**

To apply orthogonality of Lamb waves of equation (16), velocity fields \( v_x^s \) and \( v_y^s \) and stresses \( T_{xx}^s \) and \( T_{xy}^s \)

\[
T_{xx}^s = i\mu B_n \left( \left( \frac{\lambda + 2\mu}{\nu} \xi_{Sn}^2 + \lambda \xi_{Sn}^2 \right) \cos \alpha_{Sn} y - 2R_{Sn} \xi_{Sn} \beta_{Sn} \cos \beta_{Sn} y \right) e^{-i(\xi_{Sn} x - \omega t)}
\]

\[
T_{xy}^s = i\mu B_n \left( \left( \frac{\lambda + 2\mu}{\nu} \xi_{Sn}^2 + \lambda \xi_{Sn}^2 \right) \sin \alpha_{Sn} y + 2R_{Sn} \xi_{Sn} \beta_{Sn} \sin \beta_{Sn} y \right) e^{-i(\xi_{Sn} x - \omega t)}
\]

\[
T_{y}^s = \mu B_n \left[ 2\xi_{Sn} \alpha_{Sn} \sin \alpha_{Sn} y + R_{Sn} \left( \xi_{Sn}^2 - \beta_{Sn}^2 \right) \cos \beta_{Sn} y \right] e^{-i(\xi_{Sn} x - \omega t)}
\]

are required. In addition, stresses are needed to evaluate potential energy and wave power. From elasticity

Equations (19) can be rearranged using the relations

\[
(\lambda + 2\mu)\xi^2 + (\lambda + \mu)\alpha^2 = \mu(\xi^2 + \beta^2)
\]

\[
\lambda\xi^2 + (\lambda + 2\mu)\alpha^2 = -\mu(\xi^2 - \beta^2)
\]

The stresses for symmetric case become

\[
S_{xx}^s = iA_n \left[ \xi_{Sn} \sin \alpha_{Sn} y - R_{Sn} \xi_{Sn} \beta_{Sn} \sin \beta_{Sn} y \right] e^{-i(\xi_{Sn} x - \omega t)}
\]

\[
S_{yy}^s = iA_n \left[ \xi_{Sn} \sin \alpha_{Sn} y + R_{Sn} \xi_{Sn} \beta_{Sn} \sin \beta_{Sn} y \right] e^{-i(\xi_{Sn} x - \omega t)}
\]

\[
2S_{xy}^s = -A_n \left( 2\xi_{Sn} \alpha_{Sn} \cos \alpha_{Sn} y + R_{Sn} \left( \xi_{Sn}^2 - \beta_{Sn}^2 \right) \cos \beta_{Sn} y \right) e^{-i(\xi_{Sn} x - \omega t)}
\]

Similarly for antisymmetric waves, \((\text{superscript } A)\) for the sake of completeness

Velocity fields are evaluated by taking time derivative of displacements in equations (3) and (4).

\[
v_x^s = \frac{\partial u_x^s}{\partial t} = -i\omega B(\xi_S \cos \alpha_{SY} - R_S \beta_S \cos \beta_{SY}) e^{-i(\xi_S x - \omega t)}
\]

\[
v_y^s = \frac{\partial u_y^s}{\partial t} = -i\omega B(\alpha_S \sin \alpha_{SY} + R_S \xi_S \sin \beta_{SY}) e^{-i(\xi_S x - \omega t)}
\]

\[
v_y^s = \frac{\partial u_y^s}{\partial t} = i\omega A(\alpha_A \cos \alpha_{AY} + \xi_A R_A \cos \beta_{AY}) e^{-i(\xi_A x - \omega t)}
\]

Substituting equations (21) and (23) to (25) in equation (16) and performing the integration yields
\[ P_{nn} = \frac{\alpha \mu B_n^2}{2} W_S \text{ (symmetric)}, \]
\[ P_{nn} = \frac{\alpha \mu A_n^2}{2} W_A \text{ (antisymmetric)} \tag{26} \]

where
\[
W_S = \left[ \xi_d (R_s^2 + 1) (\xi_s^2 + \beta_s^2) - R_s^2 \xi_s (\xi_s^2 - 3 \beta_s^2) \frac{\sin \beta_s d \cos \beta_s d}{\beta_s} \right] \\
+ \xi (\xi_s^2 + \beta_s^2 - 4 \alpha_s^2) \frac{\sin \alpha_s d \cos \alpha_s d}{\alpha_s} \\
+ 4 R_s \xi \alpha_s \beta_s \sin \alpha_s d \cos \beta_s d - 2 R_s (3 \xi_s^2 + \beta_s^2) \cos \alpha_s d \sin \beta_s d \\
+ 4 R_s \xi \alpha_s \beta_s \sin \alpha_s d \cos \beta_s d - 2 R_s (3 \xi_s^2 + \beta_s^2) \sin \alpha_s d \cos \beta_s d \\
\] \tag{27}

\[
W_A = \left[ - \xi_d (R_s^2 + 1) (\xi_s^2 + \beta_s^2) - R_s^2 \xi_s (\xi_s^2 - 3 \beta_s^2) \frac{\sin \beta_s d \cos \beta_s d}{\beta_s} \right] \\
+ \xi (\xi_s^2 + \beta_s^2 - 4 \alpha_s^2) \frac{\sin \alpha_s d \cos \alpha_s d}{\alpha_s} \\
+ 4 R_s \xi \alpha_s \beta_s \cos \alpha_s d \cos \beta_s d - 2 R_s (3 \xi_s^2 + \beta_s^2) \sin \alpha_s d \sin \beta_s d \\
+ 4 R_s \xi \alpha_s \beta_s \cos \alpha_s d \cos \beta_s d - 2 R_s (3 \xi_s^2 + \beta_s^2) \sin \alpha_s d \sin \beta_s d \right] \\
\] \tag{28}

The symmetric mode coefficient \( B_s \) and the antisymmetric mode coefficient \( A_n \) can be resolved as
\[
B_n = \sqrt{\frac{2P_{nn}}{\alpha \mu W_{Sn}}} \text{ (symmetric)}, \quad A_n = \sqrt{\frac{2P_{nn}}{\alpha \mu W_{An}}} \text{ (antisymmetric)} \tag{29} \]

For normal modes, we may assume \( P_{nn} = 1 \); hence
\[
B_n = \sqrt{\frac{2}{\alpha \mu W_{Sn}}} \text{ (symmetric)}, \quad A_n = \sqrt{\frac{2}{\alpha \mu W_{An}}} \text{ (antisymmetric)} \tag{30} \]

**Modal contribution factors and PWAS excitation**

Basic assumptions used in this study are (1) straight-crested Lamb waves, that is, \( z \)-invariant and (2) ideal bonding (pin-force) connection between PWAS and structure (Figure 4). After consideration of the orthogonality of Lamb wave modes and after the normalization of mode amplitudes with respect to the power, modal participation factor for each mode need to be evaluated (i.e. how much a particular mode contributes to the total wave power and energy). This uses the reciprocity relation with consideration of excitation forces from the source (e.g. a PWAS on the excited structure).

Recalling the complex reciprocity equation (7), multiplying by \(-1\), and upon expansion of the del operator, we get

\[
\frac{\partial}{\partial y} (\tilde{v}_2 \cdot T_1 - v_1 \cdot \tilde{T}_2) \cdot \hat{y} + \frac{\partial}{\partial x} (\tilde{v}_2 \cdot T_1 - v_1 \cdot \tilde{T}_2) = \tilde{x} = \tilde{v}_1 F_1 + v_1 \tilde{F}_2 \tag{31} \]

where \( F \) is a volume source, \( T \cdot \hat{y} \) are the traction forces, and \( v \) are the velocity sources. Solution denoted by “1” like \( \tilde{v} \) indicates traction due to source excitation (e.g. by PWAS), while solution denoted by “2” is representing normal modes, that is, homogeneous solution of eigenfunctions of the free mode shapes of the structure—without considering excitation from the source. Fields due to excitation source can be represented as normal mode summation over all possible modes (Rose, 1999; Santoni, 2010), that is

\[
v_1 = v_1(x,y) = \sum_m a_m(x)v_m(y) \]
\[
T_1 = T_1(x,y) = \sum_m a_m(x)T_m(y) \tag{32} \]

where \( a_m(x) \) are the modal participation factors that must be determined.

Homogeneous solution “2” can be represented as

\[
v_2(x,y) = v_2(y) e^{-ik_2 x} \]
\[
T_2(x,y) = T_2(y) e^{-ik_2 x} \tag{33} \]

Integrate equation (31) with respect to the plate thickness \( y \) from \( y = -d \) to \( y = +d \) to get

\[
\left( \tilde{v}_2 \cdot T_1 - v_1 \cdot \tilde{T}_2 \right) \cdot \hat{y} \bigg|_{-d}^{+d} \\
+ \int_{-d}^{d} \frac{\partial}{\partial x} (\tilde{v}_2 \cdot T_1 - v_1 \cdot \tilde{T}_2) \cdot \hat{x} \, dy = \int_{-d}^{d} \tilde{v}_1 F_1 \, dy \tag{34} \]

**Figure 4.** Pin-force model for structurally bonded PWAS: (a) PWAS pin forces at the ends on the upper surface and (b) shear stresses developed.

Substitution of equations (32) and (33) and rearrangement yields

\[
(-\mathbf{v}_n(y) \cdot \mathbf{T}_1 - v_1 \cdot \mathbf{T}_n(y)) \cdot \mathbf{\hat{s}}|^{\sigma}_{\eta} e^{i\xi_n} \\
+ \frac{\partial}{\partial x} \left[ e^{i\xi_n} \sum_n a_n(x) \left( (-\mathbf{v}_n(y) \cdot \mathbf{T}_n(y) - v_n(y) \cdot \mathbf{T}_n(y)) \cdot \mathbf{\hat{s}} \right) dy \right] \\
= e^{i\xi_n} \int \mathbf{v}_n(y) \mathbf{F}_1 \ dy
\]

(35)

Recall the orthogonality relation in its general form

\[
P_{mn} = -\frac{1}{d} \int_{-d}^{d} (-\mathbf{v}_n(y) \cdot \mathbf{T}_m(y) + v_n(y) \cdot \mathbf{T}_n(y)) \cdot \mathbf{\hat{s}} \ dy
\]

(36)

In the absence of a volume force source term \( \mathbf{F}_1 \), equation (35) yields

\[
(-\mathbf{v}_n \cdot \mathbf{T}_1 - v_1 \cdot \mathbf{T}_n) \cdot \mathbf{\hat{s}}|^{\sigma}_{\eta} e^{i\xi_n} + \frac{\partial}{\partial x} e^{i\xi_n} \sum_n 4a_n(x)P_{mn} = 0
\]

(37)

Since the modes are orthogonal, the summation in equation (37) has only one nonzero term corresponding to the propagating mode \( n (\xi_n \text{ real}) \) for which \( P_{mn} \neq 0 \). Hence, equation (37) becomes

\[
4P_{mn} \left( \frac{\partial}{\partial x} + i\xi_n \right) a_n(x) = (\mathbf{v}_n \cdot \mathbf{T}_1 + v_1 \cdot \mathbf{T}_n) \cdot \mathbf{\hat{s}}|^{\sigma}_{\eta} e^{i\xi_n}
\]

(38)

This is a general ordinary differential equation (ODE), which needs to be solved to get the modal participation factor \( a_n(x) \). \( \mathbf{T}_n \) is the traction force; it must satisfy the traction-free boundary condition for Lamb waves, \( T_{xy} \bigg|_{z=0} = 0 \) and \( T_{xy} \bigg|_{z=d} = 0 \). \( T_1 \) is the excitation shear.

We have \( T_{xy} \big|_{d} = t_s(x) \) at the upper surface and \( T_{xy} \big|_{-d} = 0 \) on the lower surface since PWAS excitation is only on the upper surface (Figure 4).

For Lamb waves, equation (38) takes the form

\[
4P_{mn} \left( \frac{\partial}{\partial x} + i\xi_n \right) a_n(x) = v_1^y T_{yy}^n(y) + v_1^y T_{yy}^n(y) d e^{i\xi_n} \bigg|_{-d}^{d}
\]

(39)

Applying traction-free conditions and PWAS excitation, and then solving the ODE, yields

\[
a_n^+ (x) = \left[ \frac{\tilde{v}_n^x(d)}{4P_{mn}} \int_{-a}^{a} e^{i\xi_n t_s(x)} d\xi \right] e^{-i\xi_n x} \text{ for } x > a
\]

(40)

It should be noted that this formula is only for forward wave solution and outside the excitation region, that is, for \( x > a \).

The total particle velocity using NME can be written as

\[
v(x,y) = \sum_n a_n(x)v_n(y)
\]

(41)

where \( v_n(y) \) is the velocity mode shape of the \( n \)th mode, that is, \( v_n(y) = \{\tilde{v}_n^x(y)\} \). \( v_n(y) \) can be derived using the combination of the symmetric particle velocity in equation (24) with symmetric normalization coefficient of equation (30) and antisymmetric particle velocity in equation (25) with antisymmetric normalization coefficient of equation (30).

We exemplify the NME method for velocity fields with two examples: (1) 1-mm-thick aluminum plate, up to 2000 kHz where only \( S_0 \) and \( A_0 \) modes exist and (2) 2.6-mm-thick steel plate, with excitation up to 500 kHz. Figure 5 shows the particle velocity at plate’s surface in \( x \)-axis and \( y \)-axis for the two plates. Note that the values of NME velocities are not multiplied yet by PWAS excitation. The displayed results are only for first symmetric \( S_0 \) and antisymmetric \( A_0 \) modes; the multimode demonstration will be shown in a later section.

Considering the ideal bonding assumption (pin-force model), the load transfer takes place over an infinitesimal region at the PWAS end. Assuming a PWAS with center at \( x_0 = 0 \) and length \( L = 2a \), the traction on the plate surface can be written as

\[
t_s(x) = a_PF_0[\delta(x - a) - \delta(x + a)]
\]

(42)

Here, \( P_0 \) is the pin-force per unit width. Substitution of equation (42) into equation (40) gives the mode participation factor under PWAS excitation as

\[
\alpha_n^{PWAS}(x) = \frac{\tilde{v}_n^x(d)}{4P_{mn}} F_0(\omega) [e^{i\xi_n} - e^{-i\xi_n}] e^{-i\xi_n x} = g_n F_0(\omega) e^{-i\xi_n x}
\]

(43)

where \( g_n \) is the coefficient \( g_n = (\tilde{v}_n^x(d)/4P_{mn}) [e^{i\xi_n} - e^{-i\xi_n}] \).

The Lamb wave NME of the particle displacement under PWAS excitation is

\[
u(x,y) = \frac{1}{i\omega} \sum_n g_n F_0(\omega) e^{-i\xi_n x} v_n(y)
\]

(44)

The displacement in \( x \)-direction at the PWAS end \( (x = a, y = d) \) is

\[
u_s(a,d) = \frac{1}{i\omega} \sum_n \tilde{v}_n^x(d)[e^{i\xi_n} - e^{-i\xi_n}] F_0(\omega) e^{-i\xi_n}
\]

(45)
Consider a PWAS of length $l = 2a$, width $b$, and thickness $t_a$; the relation between PWAS pin-force applied to the structure and the particle displacement is through the structural dynamic stiffness. The structures as well as the PWAS stiffness are now analyzed. When the PWAS transmitter is excited by an oscillatory voltage, its volume expands in phase with the voltage in accordance with the piezoelectric effect (Figure 6). Expansion of the PWAS mounted on the surface of the structure induces a surface reaction from the structure in the form of a force at the PWAS end. The PWAS end displacement is constrained by the plate and is equal to the plate displacement at $x = a$. The reaction force along the PWAS edge, $F_0(\omega)b$, depends on the PWAS displacement, $u_{PWAS}$, and on the frequency-dependent dynamic stiffness, $k_{str}(\omega)$, presented by the structure to the PWAS.

$$F_0(\omega)b = k_{str}(\omega)u_x(a, d)$$  \hspace{1cm} (46)$$

The two stiffness elements on the right and the left of PWAS are selected to be $2k_{str}$; hence, the overall structure stiffness is $k_{str}$ (Figure 6). Under harmonic excitation, the dynamic stiffness $k_{str}(\omega)$ is obtained by dividing the force by the displacement given by equation (44), that is...
The relation between pin-force per unit width and the static stiffness of the PWAS is

$$k_{str}(\omega) = \frac{F_0(\omega) b}{u_s(a, d)} = i\omega b \left[ \sum_{n} g_n e^{-i\phi_n} y_n(d) \right]^{-1}$$  \hspace{1cm} (47)

Define the static stiffness $k_{PWAS}$ of a free PWAS as

$$k_{PWAS} = \frac{t_e b}{s_{11} a}$$  \hspace{1cm} (48)

The dynamic stiffness ratio is defined as the ratio between $k_{str}(\omega)$ and $k_{PWAS}$, that is

$$r(\omega) = \frac{k_{str}(\omega)}{k_{PWAS}}$$  \hspace{1cm} (49)

The relation between pin-force per unit width and the static stiffness of the PWAS is

$$F_0(\omega)b = k_{PWAS}\left[ u_s(a, d) - \frac{1}{2} u_{ISA} \right]$$  \hspace{1cm} (50)

where $u_{ISA}$ is the “induced strain actuation” displacement (Giurgiutiu, 2008), which is defined as $u_{ISA} = l d_{31} V / t_s$, and the quantity $u_s(a, d) - \frac{1}{2} u_{ISA}$ represents the total $x$-direction displacement at the right tip of the PWAS (because of symmetry, only forward propagating wave needs to be considered.)

Substitute $u_s(a, d)$ from equation (45) into equation (50) and solve for $F_0$ using $k_{str}$ from equation (47) to get

$$F_0(\omega) = \frac{1}{b} \left[ \frac{r(\omega)}{1 - r(\omega)} \right] \frac{k_{PWAS} u_{ISA}}{2}$$  \hspace{1cm} (51)

The excitation pin-force $F_0(\omega)$ can now be used to determine the NME fields (displacements–strains–velocities), the modal participation factors $a_n^{PWAS}(\chi)$, and the coefficients $g_n$, then the power and energy can be analyzed.

**Power transduction between PWAS and structure**

The power and energy that activates the expansion–contraction motion of the PWAS transducer. This motion is transmitted to the underlying structure through the shear stress in the adhesive layer at the PWAS–structure interface. As a result, ultrasonic guided waves are excited into the underlying structure. The mechanical power at the interface becomes the acoustic wave power, and the generated Lamb waves propagate in the structure.

**PWAS admittance and electrical active power**

To calculate the transmitter electrical power and energy, we need to calculate the input electrical power by using input admittance of the PWAS when attached to the structure. Because of the electromechanical coupling, the impedance is strongly influenced by the dynamic behavior of the structure and is substantially different from the free-PWAS impedance.

Under harmonic excitation, the time-averaged power is the average amount of energy converted per unit time under continuous harmonic excitation. The time-averaged product of the two harmonic variables is one half the product of one variable times the conjugate of the other. When a harmonic voltage is applied to the transmitter PWAS, the current is

$$I = Y V$$  \hspace{1cm} (52)

The constrained PWAS admittance can be expressed (Giurgiutiu, 2008) using the frequency-dependent stiffness ratio of equation (49), that is

$$Y(\omega) = i\omega C_0 \left[ 1 - k_{31}^2 \left( 1 - \frac{1}{r(\omega)} + \phi(\omega) \cot \phi(\omega) \right) \right]$$  \hspace{1cm} (53)

where $\phi(\omega) = \xi(\omega) a$. A simplified form of equation (53) can be obtained under the quasi-static assumption in which the PWAS dynamics are assumed to happen at much higher frequencies than the Lamb wave propagation ($\phi(\omega) \to 0, \phi(\omega) \cot \phi(\omega) \to 1$), that is

$$Y(\omega) = i\omega C_0 \left[ 1 - k_{31}^2 \frac{r(\omega)}{1 + r(\omega)} \right]$$  \hspace{1cm} (54)
This simplified model of admittance was used in Lin and Giurgiutiu (2012); it was used for axial and flexural wave propagation at low-frequency excitation. Here, we use a new definition of \( r(\omega) \) in equation (49) and \( k_{NR}(\omega) \) in equation (47) based on NME for multimodal Lamb wave propagation.

The power rating, time-averaged active power, and reactive power are

\[
P_{\text{rating}} = \frac{1}{2} |Y| \dot{V}^2 = \sqrt{P_{\text{active}}^2 + P_{\text{reactive}}^2},
\]
\[
P_{\text{active}} = \frac{1}{2} Y_R \dot{V}^2, \quad P_{\text{reactive}} = \frac{1}{2} Y_I \dot{V}^2
\]

where \( Y_R \) is the real part of admittance and \( Y_I \) is the imaginary part of admittance.

The active power is the power that is converted to the mechanical power at the interface. The reactive power is the imaginary part of the complex power that is not consumed and is recirculated to the power supply. The power rating is the power requirement of the power supply without distortion. In induced strain transmitter applications, the reactive power is the dominant factor since the transmitter impedance is dominated by its capacitive behavior (Lin et al., 2012). Managing high reactive power requirements is one of the challenges of using piezoelectric induced strain actuators.

**Mechanical power**

Due to electromechanical transduction in the PWAS, the electrical active power is converted into mechanical power, and through shear effects in the adhesive layer between PWAS and the structure, the mechanical power transfers into the structure and excites guided wave. Santoni (2010) studied shear lag solution for the case of multiple Lamb wave modes. This solution can be simplified by considering that the shear stress transfer is concentrated over some infinitesimal distances at the ends of the PWAS actuator (Figure 4). The concept of ideal bonding (also known as the pin-force model) assumed that all the load transfer takes place over an infinitesimal region at the ends of the PWAS actuator (Figure 4). The time-averaged wave power can be determined from equations (17), (24), and (25). Figure 8 shows all associated stresses and velocities. The time-averaged wave power is

\[
\langle p \rangle = \frac{1}{2} \int_0^T \dot{F}_0(\omega) \dot{v}_0(\omega) dA
\]

where \( \delta \) denotes particle velocity either in \( x \)- or \( y \)-direction, \( \dot{T} \) denotes stress, and \( \tilde{T} \) is the conjugate, which are determined from equations (17), (24), and (25). Equation (58) shows all associated stresses and velocities.

The time-averaged wave power can be determined for a given section \( x \) by integration over the cross-sectional area. Under the \( z \)-invariant assumption, the width \( b \) is taken outside the integration; equation (58) can be further simplified as

\[
\langle p \rangle = -\frac{b}{2} \int_{-d}^{d} \left\{ 2 \mu (\lambda + 2\mu) S_{xx} \right\} v_x + (2\mu S_{yy}) v_y \right\} dy
\]

Orthogonality of Lamb waves can be used during the expansion of equation (59) because all the quantities are defined as summation of symmetric solution plus
antisymmetric solution, for example, $v_n$ is the summation of $v_p$ parts of equations (24) and (25). The same is true for strains.

When evaluating the multiplication of $\tilde{S}_{ae}$ times $v_n$ and integrating, the quantities generated from multiplying symmetric part times antisymmetric parts end up as integration of sine times cosine terms and vanish due to orthogonality of Lamb waves. However, “cos$^2$” and “sin$^2$” terms are retained. The time-averaged wave power takes the closed form

$$\langle p \rangle = \sum_n \langle p_n^S \rangle + \sum_n \langle p_n^A \rangle$$  \hspace{1cm} (60)

where $\langle p_n^S \rangle$ and $\langle p_n^A \rangle$ are the time-averaged wave powers for symmetric mode $S_n$ and antisymmetric mode $A_n$, respectively.

The time-averaged kinetic energy associated with symmetric and antisymmetric modes.

$$\langle k_c \rangle = \sum_n \langle k_n^S \rangle + \sum_n \langle k_n^A \rangle$$  \hspace{1cm} (65)

where $\langle k_n^S \rangle$ and $\langle k_n^A \rangle$ are the time-averaged kinetic energies for symmetric mode $S_n$ and antisymmetric mode $A_n$. Upon multiplication and then integration over thickness, kinetic energy can be expressed in closed form as

$$\langle k_n^S \rangle = -\frac{b}{4} \left[ g_n^S B_n F(\omega) \right]^2$$

$$\langle k_n^A \rangle = -\frac{b}{4} \left[ g_n^A A_n F(\omega) \right]^2$$

$$\text{Equations (66) and (67) can be further simplified as}$$

$$\langle k_n^S \rangle = -\frac{b}{4} \left[ g_n^S B_n F(\omega) \right]^2$$

$$\langle k_n^A \rangle = -\frac{b}{4} \left[ g_n^A A_n F(\omega) \right]^2$$

Potential energy of the wave can be evaluated by the double inner (double dots) product between stress and strain

$$v_n(x, t) = \frac{1}{2} \int A \left( \tau_{xx} \dot{v}_x + \tau_{yy} \dot{v}_y \right) dA$$

$$v_n(x, t) = \frac{1}{2} \int A \left( T_{xx} T_{xy} T_{xz} T_{yy} T_{yz} T_{zz} \right) \left( S_{xx} S_{xy} S_{xz} S_{yy} S_{yz} S_{zz} \right) dA$$

$T_{zz}$ and $S_{zz}$ are ignored due to the $z$-invariant assumption. Also, $T_{yx} = T_{xy}, T_{xz} = T_{zx},$ and $T_{yz} = T_{zy}$ due to symmetry of both stress and strain tensors; equation (71) yields
\[ v_e(x,t) = \frac{1}{2} \int_A \left( T_{xx}S_{xx} + T_{yy}S_{yy} + 2T_{xy}S_{xy} + 2T_{xx}S_{xx} + 2T_{zz}S_{zz} \right) dA \] (72)

Stresses and strains associated with Lamb waves are \( T_{xx}, T_{yy}, \) and \( T_{xy} \) and \( S_{xx}, S_{yy}, \) and \( S_{xy}, \) respectively, and then Lamb wave potential energy reduces to

\[ v_e(x,t) = \frac{1}{2} \int_A \left( T_{xx}S_{xx} + T_{yy}S_{yy} + 2T_{xy}S_{xy} \right) dA \] (73)

The time-averaged potential energy is

\[ \langle v_e \rangle = \frac{1}{2} \int_A \left\{ \lambda + 2\mu \right\} S_{xx} \bar{S}_{xx} + 2\lambda S_{yy} \bar{S}_{xx} + \lambda + 2\mu \right\} S_{yy} \bar{S}_{yy} + 2\left( 2\mu S_{xy} \right) \bar{S}_{xy} \} dA \] (74)

Similar to the kinetic energy, the time-averaged potential energy is the summation of the potential energy of all modes, that is

\[ \langle v_e \rangle = \sum_n \langle v_e^n \rangle \] (75)

where \( \langle v_e^n \rangle \) and \( \langle v_e^s \rangle \) are the time-averaged potential energies for symmetric mode \( S_n \) and antisymmetric mode \( A_n, \) respectively, that is

\[ \langle v_e^n \rangle = \frac{b}{4} [g_{nn} B_n F_0(\omega)]^2 \]

\[ \left( \lambda + 2\mu \right) \int_{-d}^{d} (\xi_{a_n} \cos \alpha_{a_n} - R_{a_n} \beta_{a_n} \cos \beta_{a_n})^2 dy + \]

\[ \left( 2\lambda \right) \int_{-d}^{d} \frac{d}{-d} \left( \xi_{a_n} \cos \alpha_{a_n} - R_{a_n} \beta_{a_n} \cos \beta_{a_n} \right) \cos \alpha_{a_n} \cos \beta_{a_n} dy + \]

\[ \left( \lambda + 2\mu \right) \int_{-d}^{d} \left( \alpha_{a_n} \cos \alpha_{a_n} + R_{a_n} \beta_{a_n} \cos \beta_{a_n} \right) \cos \alpha_{a_n} \cos \beta_{a_n} dy + \]

\[ \mu \int_{-d}^{d} \left( 2\xi_{a_n} \sin \alpha_{a_n} \sin \beta_{a_n} + R_{a_n} (\xi_{a_n} - \beta_{a_n}) \sin \beta_{a_n} \right)^2 dy \] (76)

\[ \langle v_e^s \rangle = \frac{b}{4} [g_{ss} A_n F_0(\omega)]^2 \]

\[ \left( \lambda + 2\mu \right) \int_{-d}^{d} (\xi_{a_n} \sin \alpha_{a_n} - R_{a_n} \beta_{a_n} \sin \beta_{a_n})^2 dy + \]

\[ \left( 2\lambda \right) \int_{-d}^{d} \frac{d}{-d} \left( \xi_{a_n} \sin \alpha_{a_n} - R_{a_n} \beta_{a_n} \sin \beta_{a_n} \right) \sin \alpha_{a_n} \sin \beta_{a_n} dy + \]

\[ \left( \lambda + 2\mu \right) \int_{-d}^{d} \left( \alpha_{a_n} \sin \alpha_{a_n} + R_{a_n} \beta_{a_n} \sin \beta_{a_n} \right) \sin \alpha_{a_n} \sin \beta_{a_n} dy + \]

\[ \mu \int_{-d}^{d} \left( 2\xi_{a_n} \cos \alpha_{a_n} + R_{a_n} (\xi_{a_n} - \beta_{a_n}) \cos \beta_{a_n} \right)^2 dy \] (77)

The total time-averaged Lamb wave energy at the plate cross-section corresponding to the PWAS end is

\[ \langle e_e \rangle = \langle k_e \rangle + \langle v_e \rangle \] (78)

**Simulation results**

This section gives results of the simulation of power and energy transduction between PWAS and structure using the exact Lamb wave model. Comparison is performed between the exact Lamb wave model results presented here and the simplified axial and flexural wave model results of Lin and Giurgiutiu (2012). This is followed by a parametric study to show how wave power and energy change with different PWAS size and excitation frequency. The last part of this section will show the applicability of our model for the case of multimodal Lamb waves, which happen either at higher frequency or in thicker structures. We exemplify with simulation of two plates: (1) 1-mm-thick aluminum plate up to 2000 kHz and (2) 12.7-mm-thick steel plate \( (V_2) \) in up to 500 kHz frequency. Figure 9(a) and (b) shows dispersion curves for the two plates. Figure 9(a) also shows how the simplified axial and flexural waves are compared with \( S_0 \) and \( A_0 \) Lamb waves at low frequencies.

For 1-mm-thick aluminum plate, harmonic excitation of 10 V is applied on a 7 mm PWAS with frequency sweep from 1 to 2000 kHz such that only \( S_0 \) and \( A_0 \) Lamb waves exist. However, the 12.7-mm-thick steel plate (Figure 9(b)) is excited up to 500 kHz, such that three symmetric modes \( (S_0, S_1, S_2) \) and three antisymmetric modes \( (A_0, A_1, A_2) \) exist. Complete simulation parameters are given in Tables 1 and 2.

**Thin plate structure (one symmetric and one antisymmetric mode)**

The simulation results for the 1-mm-thick aluminum structure are given in Figure 10. As expected, the reactive electrical power required for PWAS excitation is orders of magnitude larger than the active electrical power. Hence, the power rating of the PWAS transmitter is dominated by the reactive power, that is, by the capacitive behavior of the PWAS. We note that the transmitter reactive power is directly proportional to the transmitter admittance \( Y = i\omega C \), whereas the transmitter active power is the power converted into the ultrasonic acoustic waves generated into the structure from the transmitter. A remarkable variation of active power with frequency is shown in Figure 10(a); we notice that the active power (i.e. the power converted into the ultrasonic waves) is not monotonic with frequency but manifests peaks and valleys. As a result, the ratio between the reactive and active powers is not constant but presents the peaks and valleys. The increase and decrease of active power with frequency correspond
to the PWAS tuning in and out of various ultrasonic waves traveling into the structure. The maximum active power seems to be 8 mW at 340 kHz. At 1460 kHz, PWAS is not transmitting any power; hence, no power was delivered into wave power at this frequency. This is because of tuned rejection Lamb waves at this particular frequency for both $S_0$ and $A_0$. Figure 5(a) and (b) shows that $v_x$ and $v_y$ vanish for both $S_0$ and $A_0$ at ~1460 kHz. Since the electrical active power is equally divided into forward and backward waves, the Lamb wave power plot of Figure 10(c) is the half of the electrical active power plot of Figure 10(a). Figure 10(d) shows the simulation results for Lamb wave kinetic energies (equations (68) and (69)) and potential energies (equations (76) and (77)).

**Comparison with low-frequency (axial and flexural) approximation**

Figure 9(a) shows that axial wave can be approximated to $S_0$ mode up to ~700 kHz for this particular case of excited 1-mm-thick aluminum plate. Flexural wave can approximate to $A_0$ for up to ~100 kHz. The axial and flexural models of Lin and Giurgiutiu (2012) compared with our exact Lamb wave model show good agreement at relatively low-frequency excitation (Figure 11). At higher frequencies, that is, beyond 700 kHz for $S_0$ and 100 kHz for $A_0$, the differences between exact and approximate models are very significant.

**Parametric study**

Figure 12 presents the results of a parametric study for various PWAS sizes (5–25 mm) and frequencies (1–1000 kHz). The resulting parametric plots are presented

---

**Table 1.** Structure simulation parameters.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>2024 Al alloy</th>
<th>AISI 4340 steel</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length $L$</td>
<td>80 mm</td>
<td>80 mm</td>
</tr>
<tr>
<td>Thickness $h$</td>
<td>1 mm</td>
<td>12.7 mm (½ in)</td>
</tr>
<tr>
<td>Width $b$</td>
<td>7 mm</td>
<td>7 mm</td>
</tr>
<tr>
<td>Young’s modulus $E$</td>
<td>72.4 GPa</td>
<td>200 GPa</td>
</tr>
<tr>
<td>Poisson ratio $\nu$</td>
<td>0.33</td>
<td>0.39</td>
</tr>
<tr>
<td>Density $\rho$</td>
<td>2780</td>
<td>7850</td>
</tr>
<tr>
<td>Harmonic input voltage amplitude $V$</td>
<td>10 V</td>
<td>10 V</td>
</tr>
<tr>
<td>Frequency $F$</td>
<td>Sweep: 1–2000 kHz</td>
<td>1–500 kHz</td>
</tr>
</tbody>
</table>

**Table 2.** Transmitter PWAS (PZT850) properties (as from the company website www.americanpiezo.com).

<table>
<thead>
<tr>
<th>Symbol</th>
<th>PZT850</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length $l$</td>
<td>5–25 mm</td>
</tr>
<tr>
<td>Thickness $t_a$</td>
<td>0.2 mm</td>
</tr>
<tr>
<td>Width $b$</td>
<td>7 mm</td>
</tr>
<tr>
<td>Young’s modulus $E$</td>
<td>63 GPa</td>
</tr>
<tr>
<td>Elastic compliance $s_E$</td>
<td>$15.8 \times 10^{-12}$ m²/N</td>
</tr>
<tr>
<td>Relative dielectric constant $\varepsilon_{33}/\varepsilon_0$</td>
<td>1750</td>
</tr>
<tr>
<td>Coupling coefficient $k_{31}$</td>
<td>0.353</td>
</tr>
<tr>
<td>Piezoelectric coefficient $d_{31}$</td>
<td>$-175 \times 10^{-12}$ m/V</td>
</tr>
</tbody>
</table>

PWAS: piezoelectric wafer active sensor.

---

**Figure 9.** Dispersion curves: (a) 1-mm-thick aluminum plate and (b) 12.7-mm-thick steel plate (½ in).

---

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Figure 10. Simulation results for 1-mm-thick aluminum plate: (a) electrical active power, (b) electrical reactive power, (c) Lamb wave power, and (d) Lamb wave kinetic and potential energies.

Figure 11. Comparison between (a) axial and S0 wave powers, (b) flexural and A0 wave powers.
as three-dimensional (3D) mesh plots. It indicates that active power generated from the PWAS to the structure contains the tuning effect of transmitter size and excitation frequency. A larger PWAS does not necessarily produce more wave power at a given frequency. The maximum active power in the simulation is 8.3 mW. This can be achieved by different combinations of PWAS and excitation frequencies (e.g. 5 mm PWAS size and 610 kHz, 9 mm PWAS and 890 kHz, 20 mm PWAS and 680 kHz, or 23 mm PWAS and 820 kHz). These combinations provide guidelines for the design of transmitter size and excitation frequency in order to obtain maximum wave power into the SHM structure.

Similarly, the total Lamb wave energy shows the same tuning trend in relation of PWAS sizes and excitation frequency as shown in Figure 13.

**Thick plate structure (multimode Lamb waves)**

Multimodal Lamb wave simulation was performed on a 12.7-mm-thick steel plate for up to 500 kHz excitation frequency and 10 V harmonic voltage applied to PWAS. All structural simulation parameters are listed in Table 1 and PWAS parameters are listed in Table 2.

The objective is to evaluate electrical active power and reactive power and show how the active power part is converted to Lamb wave power in the presence of multimodal Lamb waves. Dispersion curve plots in Figure 9(b) show that $S_1$ mode starts at $\sim 215$ kHz and $S_2$ starts at $\sim 370$ kHz, while $A_1$ starts at $\sim 170$ kHz and $A_2$ starts at 390 kHz. Due to sudden appearance of Lamb wave modes at cutoff frequencies, the NME solution encounters sudden jumps due to sudden appearance of new components due to the new modes. For that reason, a smoothing function is applied as described in the next section.

**Smoothing function.** The smoothing function is a smoothed step function (Sohoni, 1995) as shown in Figure 14. Mathematical formula is

$$ f(x) = \begin{cases} h_1 & x<X_1 \\ h_2 & X_1 \leq x \leq X_2 \\ h_2 & x>X_2 \end{cases} $$

$$ f(x) = h_1 + \frac{\Delta h}{\Delta x} (x - X_1) - \frac{\Delta h}{2\pi} \sin \left( \frac{2\pi}{\Delta x} (x - X_1) \right) $$

This is implemented in our NME solution by setting $X_1$ to the cutoff frequency of selected mode and $h_1$ to 0; hence, the mode is forced to start from 0, and consequently, its contribution to the NME summation is smoothed $\Delta h = 1$; $\Delta x$ is arbitrary; we selected $\Delta x = 150$ kHz for symmetric modes and $\Delta x = 120$ kHz for antisymmetric modes. NME for velocity fields after applying smoothing are shown in Figure 15.

For the sake of clarity, it needs to be mentioned that the plots in Figure 15 are the absolute values of NME velocities after applying normalized amplitudes as well as modal participation factors. Note that the summation value in some areas is less than the individual

values because the individual values are plotted as absolute values, whereas the summation is done algebraically, which allows for some cancellations. Complex values of NME velocities are due to phase differences. Figure 16 shows the summary of the velocity fields; it displays the summation for the three symmetric modes as well as antisymmetric modes.

Figure 17 shows the reactive and active electrical power the PWAS utilizes to excite desired Lamb wave modes. It can be seen from Figure 17 that the reactive power is three orders of magnitude larger than the active power (active power is the power that is further converted to propagating wave power). The maximum active power attained in this simulation is ~0.9 mW at 500 kHz; however, if the simulation is evaluated for larger frequency sweep, active power experiences higher maximum, but careful consideration is needed as the fourth symmetric and antisymmetric modes will come into account.

Multimodal Lamb wave simulations for power are shown in Figure 18. It can be seen that maximum value for Lamb wave power is ~0.45 mW at 500 kHz, and the plot in Figure 18(a) is identical to the half of active electrical power of Figure 17(a).

Summary and conclusion

The ability to excite certain Lamb wave modes is important in SHM as different defects respond differently to various Lamb wave modes. Detection of through-thickness cracks with pulse–echo method is much better with $S_0$ mode than $A_0$ mode. While antisymmetric modes are better for detection delaminations, it disbands with pitch–catch techniques. Different
Figure 16. Summation of normal mode expanded velocities in \( x \)- and \( y \)-directions with applied modal participation factors for (a) symmetric and (b) antisymmetric modes. NME: normal mode expansion.

Figure 17. Simulation results for 12.7-mm-thick steel plate: (a) electrical active power and (b) electrical reactive power.

Figure 18. (a) Lamb wave power and (b) Lamb wave power separated as symmetric modes and antisymmetric modes.
researches studied scattering of wave energy at defects, for example, cracks; hence, it became important to develop an analytical model for multimodal Lamb wave power and wave energy. This study has analyzed the power and energy transformation from electrical to mechanical for PWAS bonded to host structures. The analysis started by reviewing of literature to show the motivation of modeling power and energy for Lamb waves for SHM applications. This was followed by the basics of Rayleigh–Lamb equation and the solution of Lamb wave symmetric and antisymmetric fields, that is, displacement strains and velocities. Power flow analysis is based on complex reciprocity and orthogonality of Lamb wave modes, and through normalization of power flow, Lamb wave displacement amplitudes were determined. The analysis for multimodal waves was based on NME technique, which was used to determine modal participation factors, that is, how much each mode contributes to the final power.

In order to calculate the transmitter electrical power and energy, we calculated the input electrical power by using input admittance of the PWAS when attached to the structure. Because of the electromechanical coupling, the impedance is strongly influenced by the dynamic behavior of the substructure and is substantially different from the free-PWAS impedance. The active power is the power that is converted to the mechanical power at the interface. The reactive power is the imaginary part of complex power that is not consumed and is recirculated to the power supply. It was shown that the reactive electrical power required for PWAS excitation is two orders of magnitude larger than the active electrical power. Hence, the power rating of the PWAS transmitter is dominated by the reactive power, that is, by the capacitive behavior of the PWAS. A remarkable variation of active power with frequency was noticed. The active power (i.e. the power converted into the ultrasonic waves) is not monotonic with frequency but manifests peaks and valleys. As a result, the increase and decrease of active power with frequency correspond to the PWAS tuning in and out of various ultrasonic waves traveling into the structure. For instance, for single symmetric and single antisymmetric excitation simulation example, there was a particular frequency at which almost there was no energy transfer for waves to propagate. Electrical active power is further divided and converted to forward propagating wave power and backward one, and our simulations were performed for only forward wave and showed that wave power was the half of electrical active power.

The developed model for Lamb wave case was compared with axial and flexural waves, which approximate Lamb waves at relatively low frequencies, and the two simulations showed good agreement. This was followed by a parametric study to optimize the transducer size with excitation frequency to guarantee maximum energy transfer between source and examined structure. In this study, it was showed that the maximum wave power can be achieved with different combinations of PWAS size and excitation frequencies. Multimodal wave simulations were presented, and this is a practical case for most on-site thick structures at which not only $S_0$ and $A_0$ modes exist when excited by PWAS. Simulations have been done for thick steel plate. The results for electrical active/reactive power and Lamb wave power are presented. This study provided a closed-form analytical model for Lamb wave power as well as kinetic and potential energies.

Future work is to validate this model through experimental study and to use the power flow normalization in finding the normalized amplitudes that best fit experimental data.

Declaration of conflicting interests

Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation, Office of Naval Research, and Air Force Office of Scientific Research.

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Appendix I

Notation

- $a$: half-length of the piezoelectric wafer transducer, m
- $a_n(x)$: modal participation factor
- $A_n$: amplitude of nth antisymmetric mode
- $A_{0,1,2}$: antisymmetric Lamb wave modes
- $b$: width, m
- $B_n$: amplitude of nth symmetric mode
- $c$: wave speed, m/s
- $c_p$: pressure (longitudinal) wave speed, m/s
- $c_s$: shear (transverse) wave speed, m/s
- $C$: mode contribution factor
- $C_0$: capacitance, $F$
- $d$: plate half thickness, m
- $d_{31}$: piezoelectric coupling coefficient in 31, m/V
- $E$: Young’s modulus, GPa
- $f$: frequency, Hz
- $F$: force vector
- $F_0(\omega)$: pin-force at PWAS ends
- $g_n$: coefficient to simplify modal participation factor
- $h$: plate thickness = 2$d$, m
- $i$: $\sqrt{-1}$
- $I$: electric current, A
- $Im$: imaginary part of a complex quantity
- $k_c$: kinetic energy
- $k_{PWAS}$: PWAS stiffness, N/m
- $k_{str}$: dynamic stiffness, N/m
- $k_{31}$: electromechanical cross-coupling coefficient
- $l_1,l_2$: PWAS length = 2$a$
- $p$: power, W
- $\langle p \rangle$: time-averaged power
- $P_{mm}$: power factor (measure of average power flow)
- $r(\omega)$: dynamic stiffness ratio
- $Re$: real part of a complex quantity
- $S_{11}$: mechanical compliance under constant electric field, m$^2$/N
- $S_{ij}$: symmetric Lamb wave modes
- $t$: time, s
- $t_a$: PWAS thickness, m
- $t_{1}(x)$: traction in x-direction
- $T_{ij}$: stress tensor for nth guided wave mode,$\text{Pa}$
- $T$: period time, s
- $T_n$: stress tensor for nth guided wave mode,
- $u$: displacement, m
- $u_{ISA}$: induced strain actuation PWAS displacement
- $U$: displacement amplitudes (also orthogonal modes), m
- $v$: velocity
- $v$: velocity vector
- $v_e$: potential energy
- $\langle v_e \rangle$: time-averaged potential energy
- $V$: voltage
- $W$: parameter to simplify normalized mode formula, m$^{-2}$
- $x, y, z$: global coordinates, m
- $X_1,X_2,X_3$: material polarization directions
- $Y$: admittance, S
- $|Y|$: absolute of admittance
- $Y_I$: imaginary part of admittance
- $Y_R$: real part of admittance
- $\alpha, \beta$: wave numbers, m$^{-1}$
- $\delta$: Kronecker delta
- $\varepsilon_{jk}$: dielectric permittivity measured at zero mechanical stress, $T = 0$
- $\lambda$: Lame constant, Pa
- $\mu$: shear modulus of the material (equivalent to the engineering constant $G$), Pa
- $\nu$: Poisson ratio
\( \xi \) wave numbers, m\(^{-1}\)
\( \rho \) material density, kg/m\(^3\)
\( \tau, \tau_d \) shear stress at PWAS tip \( x = a \)
\( \phi \) Lamb wave longitudinal potential function
\( \phi(\omega) \) dynamic
\( \psi \) Lamb wave shear potential function
\( \omega \) angular frequency, rad/s
\( \Omega \) domain

**Subscripts**

- \( A \) antisymmetric modes
- \( i, j \) indices = 1, 2, 3
- \( m, n \) different normal modes
- \( n \) \( n \)th guided wave mode
- \( S \) symmetric modes

**Superscripts**

- \( \tilde{a} \) conjugate of a
- \( \hat{a} \) amplitude of a
- \( A \) antisymmetric modes
- \( m, n \) different normal modes
- \( S \) symmetric modes
- 1 solution due to source excitation
- 2 solution due to homogeneous solution (free mode shapes)