Analysis of axis symmetric circular crested elastic wave generated during crack propagation in a plate: A Helmholtz potential technique

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A B S T R A C T
This paper presents cylindrical coordinate solutions of axis symmetric circular crested elastic waves that appear due to sudden energy release during incremental crack propagation in a plate. Axis symmetric assumption decouples the elastic wave problem to Lamb (P+SV) and shear (SH) horizontal waves. Helmholtz decomposition principle was used to decompose displacement field in to unknown scalar and vector potentials; and body force vectors to known excitation scalar and vector potentials respectively. Therefore, Navier–Lame equations yield a set of four inhomogeneous wave equations of unknown potentials \( \Phi, H_s, H_t, H_v \) and known excitation potentials \( A^r, B^r, B^s \). There are two types of potentials exist in a plate for axis symmetric circular crested Lamb wave: pressure potentials \( \Phi, A^r \) and shear potentials \( H_s, B^s \). Inhomogeneous wave equations for \( \Phi \) and \( H_s \) were solved due to generalized excitation potentials \( A^r \) and \( B^s \) in a form, suitable for numerical calculation. The theoretical formulation shows that elastic waves generated in a plate using excitation potentials follow the Rayleigh-Lamb equations. The resulting solution is a series expansion containing the superposition of all the Lamb wave modes existing for the particular frequency-thickness combination under consideration. In addition, bulk wave solution is also recovered due to the effect of the excitation potentials. The numerical studies modeled the two-dimensional (2D) (circular crested) AE elastic wave propagation in order to simulate the out-of-plane displacement that would be recorded by an AE sensor placed on the plate surface at some distance away from the source. Parameter studies were performed to evaluate: (a) the effect of the pressure and shear potentials; (b) the effect of the thickness-wise location of the excitation potential sources varying from mid-plane to the top surface (source depth effect); (c) the effect of peak time (d) the effect of propagating distance away from the source. A Gaussian pulse is used to model the growth of the excitation potentials during the AE event; as a result, the actual excitation potential follows the error function variation in the time domain. The numerical studies show that the peak amplitude of AE signal is higher than the peak amplitude of \( S_0 \) signal and the peak amplitude of bulk wave is not significant compared to \( S_0, A_0 \) peak signals.

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1. Introduction

Acoustic emission (AE) is the phenomenon of radiation of elastic waves by the rapid release of energy in a plate caused by irreversible changes in its internal structure, for example as a result of crack formation or plastic deformation. Acoustic emission (AE) techniques are therefore needed to study extensively in order to detect damage in a structure. Evaluation of AE waveform allows not only the location of the source of the emission but also the determination of its nature. Nevertheless, the use of such techniques needs a clear understanding of relationships between the elastic waves and the damage process. AE elastic waves have been used extensively in structural health monitoring (SHM) and non-destructive testing (NDT) for the detection of damage and to prevent ultimate failure (Harris and Dunegan, 1974; Han et al., 2014; Tandon and Choudhury, 1999; Roberts and Talebzadeh, 2003; Bassim et al., 1994). In this article, an attempt is made to analyze circular crested AE elastic waves theoretically and numerically by considering known potentials, which appear due to energy release from a crack.

The Helmholtz decomposition (Helmholtz, 1858) approach is applied to the inhomogeneous elastodynamic Navier–Lame equations: (a) for the displacement field, we use the usual decomposition in terms of unknown scalar and vector potentials, \( \Phi \) and \( H \); (b) for the body forces, we hypothesize that they can be also expressed in terms of potentials and we introduce the excitation scalar and vector potentials, \( A^r \) and \( B^s \). It is shown that these excitation potentials can be traced to the energy released during an incremental crack propagation. Thus, the inhomogeneous Navier–Lame
Lame equations have been transformed into a system of inhomogeneous wave equations in terms of known excitation potentials \( \Lambda^*, \bar{B}^* \) and unknown solution potentials \( \Phi, \bar{H} \). The main contribution of this paper is, therefore, to analyze AE elastic waves due to the energy released during crack propagation using the Helmholtz potential approach, which, to the best of our knowledge, has not been reported yet elsewhere.

The numerical studies modeled the two-dimensional (2D) (circular crested) AE elastic wave propagation in order to simulate the out-of-plane displacement that would be recorded by an AE sensor placed on the plate surface at some distance away from the source. A real AE source releases energy at certain time rate as a pulse over a finite time period. If the time rate of released energy is known then this time rate of released energy can be decomposed to the time rate of pressure and shear excitation potentials. The time profile of excitation potentials are then calculated from time rate of excitation potentials. A 6 mm thick stainless steel plate was chosen for the case study. Out-of-plane displacement was calculated numerically on the top surface of the plate at 500 mm distance from excitation point, while excitation point is located at mid-plane. A very short peak time (3 \( \mu s \)) of time rate of excitation potential was considered during the case study. Then, the effect of the thickness-wise location of the excitation potential sources varying from mid-plane to the top surface was considered. The peak amplitude of the \( S_0, A_0 \) and bulk wave signals as a function of propagating distance was also determined.

1.1. State of the art

Several authors (Lamb, 1917; Achenbach, 2003; Giurgiutiu, 2014; Graff, 1975; Viktorov, 1967; Landau and Lifshitz, 1965; Love, 1944; Ali and Richards, 2002) have explained the basic concepts and equations of elastodynamics. Lamb (1903) presented propagation of vibrations over the surface of semi-infinite elastic solids. The vibrations were assumed due to an arbitrary application of force at a point. Rice (1980) discussed a theory of elastic wave emission from damage processes (e.g., slip and micro-cracking). A general representation of the displacement field of an AE event was presented in terms of damage process in the source region. The solution is obtained in an unbounded medium. Ohtsu and Ono (1986) characterized the source of AE on the basis of generalized theory. AE waveforms due to point and relative motion of coordinates of instantaneously formed dislocation were presented in that paper. In an inverse study, deconvolution analysis was done to characterize the AE source. Wave propagation in an elastic plate is well known from Lamb (1917) classical work. Achenbach (2003), Giurgiutiu (2014), Graff (1975) and Viktorov (1967) considered Lamb wave, which exists in an elastic plate between two traction free boundaries. Achenbach (2003) presented Lamb wave in an isotropic elastic layer generated by a time harmonic internal/surface point load or line load. Displacements are obtained directly as summations over symmetric and antisymmetric modes of wave propagation. Elastodynamic reciprocity is used in order to obtain the coefficients in the wave mode expansion. Bai et al. (2004) presented three-dimensional steady state Green functions for Lamb wave in layered isotropic plate. The elastodynamic response of a layered isotropic plate to a source point load having an arbitrary direction was studied in that paper. A semi-analytical finite element (SAFE) method was used to formulate the governing equations. Using this method, in-plane displacements were accommodated by means of an analytical double integral Fourier transform, while, the anti-plane displacement approximated by using finite elements. They have used same modal summation technique of eigenvectors as described by Liu and Achenbach (1995), Achenbach (2003).

Jacobs et al. (1991) presented an analytical methodology by incorporating a time dependent acoustic emission signal as a source model to represent an actual crack propagation and arrest event. A reviews paper on the signal analysis used in acoustic emission (AE) as applied to materials research field published by Ono (2011). This paper reviewed recent progress in methods of signal analysis used in acoustic emission such as deformation, fracture, phase transformation, coating, film, friction, wear, corrosion and stress corrosion for materials research. A novel experimental mechanics technique using scanning electron microscopy (SEM) in conjunction with AE monitoring is discussed by Wisner et al. (2015). The objective is to investigate microstructure sensitive mechanical behavior and damage of metals, in order to validate AE related information.

There are numerous publications based on finite element analysis for detecting AE signals. For example, Hamstad et al. (1999) reported wave propagation due to buried monopole and dipole sources with finite element technique. Hill et al. (2004) compared waveforms captured by AE transducer for a step force on the plate surfaces by finite element modeling. Hamstad (2010) presented frequencies and amplitudes of AE signals on a plate as a function of source rise time. In that paper, an exponential increase in peak amplitude with source rise time was reported.

1.2. Description of excitation potentials as energy released from a crack

For incremental crack growth into a plate, a region of material adjacent to the free surfaces is unloaded, and it releases strain energy. The strain energy is the energy that must be supplied to a crack tip for it to grow and it must be balanced by the amount of energy dissipated due to the formation of new surfaces and other dissipative processes such as plasticity (Nguyen et al., 2001; Nuismer, 1975). During this incremental crack growth process, the energy release that occurs excites elastic waves in the plate. However, most of the stored energy is dissipated as heat and only a fraction of the total released energy available for AE source (Cuadra et al., 2016; Ravi-Chandar, 2004). Crack extension or crack propagation can be treated as local phenomena. In the local analysis, a convenient formulation of the J-integral can be applied to calculate energy balance (Cuadra et al., 2016). J-integral for equilibrium states provides an estimate of the total energy release, which agreed with the energy balance approach. In this case, the stress-strain relationship is essential to assign the material law in the energy balance calculation, including the elastic and plastic regimes. Once the energy release from a crack is calculated, AE elastic wave propagation can be calculated by this present approach. So far from the crack, the crack can be treated as localized AE source from where energy is released. Therefore, at some distance from the crack, the elastic waves can be assumed as axisymmetric.

Fig. 1a shows a plate with an existing crack length of 2a and Fig. 1b shows the typical strain energy release rate with crack half-length (Fischer-Cripps, 2000). Cuadra et al. (2016) proposed a 3D computational model to quantify the energy associated with AE source by energy balance and energy flux approach. The time profile of available energy as AE source was obtained for the first increment of fracture. The rate of energy release that occurs due to incremental crack growth can be different at different time depending on crack growth and types of materials. A real AE source releases energy during a finite time period. If time rate of energy released is known (Fig. 1c), then after integrating the time rate of energy with respect to time gives the time profile of energy (Fig. 1d) released from a crack. Total released energy from a crack can be decomposed to pressure and shear excitation potentials. The proceeding section presents a theoretical formulation for Lamb wave solution using time profile of excitation potentials: a Helmholtz potential technique.
1.3. Scope of this article

Various deformation sources in solids, such as single forces, step force, point force, dipoles and moments can be represented as AE sources (Ohtsu and Ono, 1986; Michals et al., 1981; Hsu et al., 1978; Pao, 1978). A moment tensor analysis to acoustic emission (AE) has been studied to elucidate crack types and orientations of AE sources (Ohtsu, 1995). AE source characteristics are unknown and the detected AE signals depend on the types of AE source, propagation media, and the sensor response. Therefore extracting AE source feature from a recorded AE waveform is always challenging. In this paper, we proposed a new technique to predict the AE waveform using excitation potentials. Excitation potentials can be traced to the energy released during an incremental crack propagation. The aim of this paper is to simulate AE elastic waves due to the energy released during crack propagation by Helmholtz potential approach.

Our analytical model is developed based on integral transform using Helmholtz potentials. The solution is obtained through direct and inverse integral transforms and application of the residue theorem. The resulting solution is a series expansion containing the superposition of all the Lamb waves modes and bulk waves existing for the particular frequency-thickness combination under consideration. It is worth mentioning that there are other methods that can be used to solve wave equation of a structure, for example, Green's functions for forced loading, semi analytical methods; the normal mode expansion method etc. Green's functions for forced loading can be solved through an integral formulation relying on the elastodynamic reciprocity principle or integral transform. However, the Green function for forced loading problem requires a point load source. However, extracting force information as an AE source from a crack is always challenging. Rather, it is easy to calculate the energy released from the tip of the crack during crack propagation. Our proposed method novelty lies in calculating AE waveform response using an energy formulation instead of specific point loads, dipoles, moment tensors, etc. used by previous investigators (Section 1.1). Our method is based on the observation that strain energy is released when internal damage takes place in the structure. However, one does not need to know the exact physics of how the damage occurs. Therefore, our potential approach is more general than other methods; our approach can address the many types of AE waveforms observed in AE practice corresponding to various source types. In this paper, we consider a particular example and show that our method can simulate the AE elastic waves generated by crack advancement. The energy released during an AE event corresponds to a combination of pressure and shear excitation potentials; the actual relative contributions of the pressure and shear potentials to the total AE energy would depend on the particular physics of the crack advancement process. Semi-analytical finite element (SAFE), boundary element method (BEM) or finite element method (FEM) requires extensive computational effort and has convergence issue at high frequency. Integral transforms provide an exact solution, and are relatively easy to solve numerically.
2. Formation of pressure and shear potentials

Navier–Lame equations for displacement in the absence of body force in vector form given as

\[(\lambda + \mu) \nabla \cdot \mathbf{u} - \mu \nabla \times \nabla \times \mathbf{u} = \rho \mathbf{u} \]  
(1)

where, \( \mathbf{u} \) is the displacement vector, \( \lambda, \mu = \) Lame constant, \( \rho = \) density.

If the body force is present, then the Navier–Lame equations can be written as follows

\[(\lambda + \mu) \nabla \cdot \mathbf{u} - \mu \nabla \times \nabla \times \mathbf{u} + \rho \mathbf{f} = \rho \mathbf{u} \]  
(2)

Helmholtz decomposition principle (Helmholtz, 1858) states that, any vector can be resolved into the sum of an irrotational (curl-free) vector field and a solenoidal vector field, where an irrotational vector field has a scalar potential and a solenoidal vector field has a vector potential. Potentials are useful and convenient in several wave theory derivations (Giurgiutiu, 2014; Graff, 1975; Aki and Richards, 2002, Bhuiyan and Giurgiutiu, 2016).

Assume that the displacement \( \mathbf{u} \) can be expressed in terms of two potential functions, a scalar potential \( \Phi \) and a vector potential \( \mathbf{H} \)

\[ \dot{\mathbf{u}} = \nabla \Phi + \nabla \times \mathbf{H} \]  
(3)

where

\[ \mathbf{H} = H_\theta \hat{e}_\theta + H_\phi \hat{e}_\phi + h_\| \hat{e}_z \]  
(4)

Eq. (3) is known as the Helmholtz equation, and is complemented by the uniqueness condition, i.e.,

\[ \nabla \cdot \mathbf{H} = f(r, \theta) \]  
(5)

Introducing additional scalar and vector potentials \( A^* \) and \( \mathbf{B}^* \) for body force \( f \) (Aki and Richards, 2002; Graff, 1991)

\[ \dot{\mathbf{f}} = \nabla A^* + \nabla \times \mathbf{B}^* \]  
(6)

Here,

\[ \mathbf{B}^* = B^*_\theta \hat{e}_\theta + B^*_\phi \hat{e}_\phi + B_z^* \hat{e}_z \]  
(7)

The uniqueness condition is,

\[ \nabla \cdot \mathbf{B}^* = f(r, \theta) \]  
(8)

Using the Eqs. (3) and (6), into Eq. (2) the wave equations for potentials are [Appendix A]

\[ c_p^2 \nabla \cdot \mathbf{H} = A^* - \Phi = 0 \]  
(9)

\[ c_l^2 \nabla \times \mathbf{H} - \mathbf{B}^* + \mathbf{H} = 0 \]  
(10)

Here, \( c_p^2 = \frac{\lambda + 2\mu}{\rho} \), \( c_l^2 = \frac{\mu}{\rho} \)

Eqs. (9) and (10) are the wave equations for the scalar potential and the vector potential respectively. Eq. (9) indicates that the scalar potential, \( \Phi \), propagates with the pressure wave speed, \( c_p \) due to excitation potential \( A^* \), whereas Eq. (10) indicates that the vector potential, \( \mathbf{H} \), propagates with the shear wave speed, \( c_l \) due to excitation potential \( \mathbf{B}^* \). The excitation potentials are indeed a true description of energy released from a crack. The unit of the excitation potentials are [J/kg]. The source can be treated as localized event and Green function can be used as the solution of inhomogeneous wave equations. In this paper, wave equations for scalar potential \( \Phi \) and vector potential \( \mathbf{H} \) need to be solved with the presence of source potentials \( A^* \) and \( \mathbf{B}^* \). The derivation Lamb wave equation will be provided by solving wave Eqs. (9) and (10) for the potentials. In this derivation, we will consider generation of axis symmetric circular crested lamb wave due to time harmonic excitation potentials. The assumption of axis-symmetric waves makes the problem \( \theta \)-invariant i.e., \( \frac{\partial}{\partial \theta} = 0 \).

Upon expansion of Eq. (2) in cylindrical coordinate, i.e.,

\[ (\lambda + \mu) \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \mathbf{r} \cdot \mathbf{u}}{\partial r} + \frac{\partial \mathbf{r} \cdot \mathbf{u}}{\partial \theta} + \frac{\partial \mathbf{u}_z}{\partial z} \right) + \mu \left( \Delta^2 \mathbf{u} - \frac{\mathbf{u}}{r^2} \right) + \rho \mathbf{f} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \]  
(11)

For axis symmetric problem, \( \frac{\partial}{\partial \theta} = 0 \), Eq. (11) becomes

\[ (\lambda + \mu) \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \mathbf{r} \cdot \mathbf{u}}{\partial r} + \frac{\partial \mathbf{r} \cdot \mathbf{u}}{\partial \theta} + \frac{\partial \mathbf{u}_z}{\partial z} \right) + \mu \left( \Delta^2 \mathbf{u} - \frac{\mathbf{u}}{r^2} \right) + \rho \mathbf{f} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \]  
(12)

Corresponding displacement and force equations from Eqs. (3) and (6), for axis symmetric condition,

\[ u_r = \frac{\partial \Phi}{\partial r} - \frac{\partial H_\theta}{\partial z} \]  
(13)

\[ u_\theta = \frac{\partial H_r}{\partial z} - \frac{\partial \Phi}{\partial \theta} \]  
(14)

\[ u_z = \frac{\partial \Phi}{\partial r} + \frac{1}{r} \frac{\partial (rH_\theta)}{\partial r} \]  
(15)

(1) Solution for \( u_r \) and \( u_\theta \), with body forces \( f_r \) and \( f_\theta \) which depend on the potentials \( \Phi, H_\theta, A^*, B^*_\phi \). The solution will be the combination of pressure wave \( P \) represented by potential \( \Phi \) and a shear vertical wave \( SV \) represented by potential \( H_\theta \) due to excitation potentials \( A^*, B^*_\phi \).

(2) Solution for \( u_\theta \) due to body force \( f_\theta \) which depend on the potentials \( H_r, H_\theta, B^*_r, B^*_z \). The solution for \( u_\theta \) displacement will give shear horizontal wave \( SH \) represented by potential \( H_r, H_\theta \) due to excitation potentials \( B^*_r, B^*_z \).

Corresponding potentials Eqs. (9) and (10) become

\[ c_p^2 \nabla \cdot \mathbf{H} + A^* = \Phi \]  
(15)

\[ c_l^2 \nabla \times \mathbf{H} - B^*_r + H_r = 0 \]  
(16)
Now for P+SV waves in a plate
\[ u_r = \frac{\partial \Phi}{\partial r} - \frac{\partial H_0}{\partial z}, \quad u_z = \frac{\partial \Phi}{\partial z} + \frac{1}{r} \frac{\partial (r H_0)}{\partial r} \] (17)
\[ f_r = \frac{\partial A^\ast}{\partial r} - \frac{\partial B_0^\ast}{\partial z}, \quad f_z = \frac{\partial A^\ast}{\partial z} + \frac{1}{r} \frac{\partial (r B_0^\ast)}{\partial r} \] (18)
\[ u_r, u_z = f(\Phi, H_0) \] (19)
\[ f_r, f_z = f(A^\ast, B_0^\ast) \] (20)

Therefore, for P+SV waves, the relevant potentials are \( \Phi, H_0, A, B_0 \). Eqs. (15) and (16) condensed to two equations, i.e.,
\[ c_p^2 \text{ div grad } \Phi + A^\ast - \Phi = 0 \] (21)
\[ c_s^2 \text{ curl curl } H_0 - B_0^\ast + \bar{H}_0 = 0 \] (22)

Upon rearranging,
\[ \text{div grad } \Phi + \frac{A^\ast}{c_p^2} - \frac{\Phi}{c_s^2} = 0 \] (23)
\[ \text{curl curl } H_0 - \frac{B_0^\ast}{c_s^2} + \bar{H}_0 = 0 \] (24)

Inhomogeneous wave equations need to be solved under zero stress boundary conditions at the top and bottom surface of the plate. The P waves and SV waves give rise to the Lamb waves, which consist of a pattern of standing waves in the thickness \( z \) direction (Lamb wave modes) behaving like traveling waves in the \( r \) direction in a plate of thickness \( h=2d \)

2.1. Source localization

Pressure and shear excitation potentials are assumed to be located at \( z_0 \) distance (Fig. 2) from the mid-plane (origin).

Excitation pressure potentials can be assumed as an axisymmetric point source (Fig. 3) emitting from location \( O(0, z_0) \) in a volume
\[ A^\ast = \frac{2}{2\pi} \frac{A \delta(r)}{r} \delta(z - z_0) \] (25)

Here \( A \) is the amplitude of the source and \( \delta \) is the Dirac delta function.

Excitation shear potential is a point source by limiting the ring source (Fig. 4) as
\[ B_0^\ast = \frac{2}{2\pi} \frac{B_0 \delta(r)}{r} \lim_{r \to 0} \frac{\delta(r-r_o)}{r_o} \delta(z - z_0) \] (26)

Here \( B_0 \) is the amplitude of the source.

3. Solutions in terms of displacement

If the source is vibrating harmonically, then equations for pressure and shear excitation potentials are
\[ A^\ast = \frac{2}{2\pi} \frac{A \delta(r)}{r} \delta(z - z_0) e^{-\omega t} \] (27)
\[ B_0^\ast = \frac{2}{2\pi} \frac{B_0 \delta(r)}{r} \lim_{r \to 0} \frac{\delta(r-r_o)}{r_o} \delta(z - z_0) e^{-\omega t} \] (28)

If the potentials \( \Phi, H_0 \), has the harmonic response then
\[ \Phi(r, z, t) = \Phi(x, y) e^{-\omega t} \] (29)
\[ H_0(r, z, t) = H_0(x, y) e^{-\omega t} \]

Eqs. (23) and (24) become
\[ \text{div grad } \Phi + \frac{\omega^2}{c_p^2} \Phi = -\frac{A}{2\pi} \frac{\delta(r)}{r} \delta(z - z_0) \] (30)
\[ \text{curl curl } H_0 - \frac{\omega^2}{c_s^2} H_0 = \frac{B_0}{2\pi} \lim_{r \to 0} \frac{\delta(r-r_o)}{r_o} \delta(z - z_0) \] (31)

In expanded form of Eqs. (30) and (31) are
\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) + \frac{\partial^2 \Phi}{\partial z^2} + \frac{\omega^2}{c_p^2} \Phi = -\frac{A}{2\pi} \frac{\delta(r)}{r} \delta(z - z_0) \] (32)
\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial H_0}{\partial r} \right) + \frac{\partial^2 H_0}{\partial z^2} + \frac{\omega^2}{c_s^2} H_0 = \frac{B_0}{2\pi} \lim_{r \to 0} \frac{\delta(r-r_o)}{r_o} \delta(z - z_0) \] (33)

Here,
\[ \text{div grad } \Phi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) + \frac{\partial^2 \Phi}{\partial z^2} \] (34)
\[ \text{curl curl } H_0 = -\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial H_0}{\partial r} \right) + \frac{\partial^2 H_0}{\partial z^2} \] (35)

Eqs. (32) and (33) must be solved to the boundary conditions
\[ \sigma_{z} |_{z=d} = 0; \quad \sigma_{z} |_{z=-d} = 0 \] (36)
Recall the Hankel transform of order \(\nu\)
\[
\hat{f}(\xi) = \int_0^\infty f(r) J_\nu(\xi r) \, dr
\]
(37)

\[f(r) = \int_0^\infty \left( \hat{f}(\xi) \right) J_\nu(\xi r) \, d\xi
\]
(38)

Here, \(f(r)\) is a generic function and \(J_\nu\) is Bessel function of order \(\nu\).

\[J_\nu(x) = \left( \frac{1}{2x} \right)^{\nu/2} \sum_{k=0}^{\infty} \frac{(-i)^k x^k}{k! \Gamma(\nu + k + 1)}
\]
(39)

Apply the Hankel transform of order 0 to Eq. (32)

\[
\int_0^\infty \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) J_0(\xi r) \, dr + \int_0^\infty \frac{\partial^2 \Phi}{\partial z^2} J_0(\xi r) \, dr
\]
\[- \int_0^\infty \frac{\omega^2}{c_p^2} \Phi J_0(\xi r) \, dr = - \int_0^\infty \frac{A}{2\pi} \frac{\delta(r-r_0)}{r} \frac{\delta(z-z_0)}{r} J_0(\xi r) \, dr
\]
(40)

Using Hankel transform identity from ref. (Giurgiutiu, 2014; pp. 608,609) and Dirac-delta identity, the integrals from Eq. (40) are

\[
\int_0^\infty \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) J_0(\xi r) \, dr = -\xi^2 \Phi

\]
(41)
\[\int_0^\infty \frac{\omega^2}{c_p^2} \Phi J_0(\xi r) \, dr = \frac{\omega^2}{c_p^2} \Phi
\]
(42)
\[\int_0^\infty \frac{\partial^2 \Phi}{\partial z^2} J_0(\xi r) \, dr = -\frac{A}{2\pi} \delta(z-z_0)
\]
(43)
\[\int_0^\infty \frac{A}{2\pi} \frac{\delta(r-r_0)}{r} \frac{\delta(z-z_0)}{r} J_0(\xi r) \, dr = \frac{A}{2\pi} \delta(z-z_0)
\]
(45)

Here, \(J_0(0) = 1\)

Therefore, Eq. (40) becomes

\[-\xi^2 \Phi + \frac{\partial^2 \Phi}{\partial z^2} + \frac{\omega^2}{c_p^2} \Phi = -\frac{A}{2\pi} \delta(z-z_0)
\]
(46)

Upon rearrangement

\[
\frac{\partial^2 \Phi}{\partial z^2} + \frac{\omega^2}{c_p^2} \Phi = -\frac{A}{2\pi} \delta(z-z_0)
\]
(47)

Apply the Hankel transform of order 1 to Eq. (33)

\[
\int_0^\infty \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \hat{H}_0}{\partial r} \right) J_1(\xi r) \, dr + \int_0^\infty \frac{\partial^2 \hat{H}_0}{\partial z^2} J_1(\xi r) \, dr
\]
\[- \int_0^\infty \frac{\omega^2}{c_p^2} \hat{H}_0 J_1(\xi r) \, dr
\]
\[- \int_0^\infty \frac{B_0}{2\pi} \lim_{r_0 \to \infty} \frac{\delta(r-r_0)}{r r_0} \frac{\delta(z-z_0)}{r_0} J_1(\xi r) \, dr
\]
(48)

Using Hankel transform identity from ref. (Giurgiutiu, 2014; pp. 609,610) and Dirac-delta identity, the integrals from Eq. (48) are

\[
\int_0^\infty \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \hat{H}_0}{\partial r} \right) J_1(\xi r) \, dr = -\xi^2 \hat{H}_0

\]
(49)
\[\int_0^\infty \frac{\partial^2 \hat{H}_0}{\partial z^2} J_1(\xi r) \, dr = \frac{\partial^2 \hat{H}_0}{\partial z^2}
\]
(50)

\[
\int_0^\infty \frac{\partial^2 \hat{H}_0}{\partial z^2} J_1(\xi r) \, dr = \frac{\partial^2 \hat{H}_0}{\partial z^2}
\]
(51)
\[\int_0^\infty \frac{B_0}{2\pi} \lim_{r_0 \to \infty} \frac{\delta(r-r_0)}{r r_0} \frac{\delta(z-z_0)}{r_0} J_1(\xi r) \, dr
\]
\[= \frac{B_0}{2\pi} \delta(z-z_0) \lim_{r_0 \to \infty} J_1(\xi r_o) = \frac{B_0}{2\pi} \delta(z-z_0) \frac{\xi}{2}
\]
(53)

Here,

\[
\lim_{r_0 \to \infty} J_1(\xi r_0) = \frac{\xi}{2}
\]

Hence, Eq. (48) becomes

\[-\xi^2 \hat{H}_0 + \frac{\partial^2 \hat{H}_0}{\partial z^2} + \frac{\omega^2}{c_p^2} \hat{H}_0 = \frac{B_0}{2\pi} \delta(z-z_0)
\]
(54)

Upon rearrangement,

\[
\frac{\partial^2 \hat{H}_0}{\partial z^2} + \frac{\omega^2}{c_p^2} \hat{H}_0 = \frac{B_0}{2\pi} \delta(z-z_0)
\]
(55)

Eqs. (47) and (55) can be simplified as,

\[
\Phi_{0y}'(\xi, y) + \frac{\omega^2}{c_p^2} \Phi_{0y}(\xi, y) = -\frac{A}{2\pi} \delta(z-z_0)
\]
(56)

\[
\tilde{H}_{0y}(\xi, y) + \frac{\omega^2}{c_p^2} \tilde{H}_{0y}(\xi, y) = -\frac{B_0}{2\pi} \frac{\xi}{2} \delta(z-z_0)
\]
(57)

Here,

\[
\left( \frac{\omega^2}{c_p^2} - \xi^2 \right) = \xi_p^2
\]
\[
\left( \frac{\omega^2}{c_s^2} - \xi^2 \right) = \xi_s^2
\]
(58)

Eqs. (56) and (57) are the second order ordinary differential equation (ODE) in \(z\) direction. The total solution of Eqs. (56) and (57) consists of the representation of two solutions (Jensen et al., 2011):

(a) The complementary solution for the homogenous equation
(b) A particular solution of that satisfies source effect from right hand side

Solution can be assumed as

\[
\Phi_{0y}(\xi, z) = \Phi_0(\xi, z) + \Phi_1(\xi, z)
\]
(59)

\[
\tilde{H}_{0y}(\xi, z) = \tilde{H}_{0y}(\tilde{H}_{0y}(\xi, z) + \tilde{H}_{0y}(\xi, z))
\]
(60)

For simplicity, subscript \(J_0\) and \(J_1\) are omitted from \(\Phi_0, \Phi_1\) and \(\tilde{H}_{0y}, \tilde{H}_{0y}\).

The functions \(\Phi_0\) and \(\tilde{H}_{0y}\) satisfy the corresponding homogeneous differential equations

\[
\Phi_{0y}'(\xi, z) + \xi_p^2 \Phi_{0y}(\xi, z) = 0
\]
(61)

\[
\tilde{H}_{0y}(\xi, z) + \xi_s^2 \tilde{H}_{0y}(\xi, z) = 0
\]
(62)

Eqs. (61) and (62) gives the complementary solutions, i.e.,

\[
\Phi_{0y}(\xi, z) = C_1 \sin \xi_p z + C_2 \cos \xi_p z
\]
(63)

\[
\tilde{H}_{0y}(\xi, z) = D_1 \sin \xi_s z + D_2 \cos \xi_s z
\]
(64)
The function \( \Phi_1(\xi, \eta) \) and \( \tilde{H}_1(\xi, \eta) \) from Eq. (60) satisfy the corresponding inhomogeneous equations

\[
\Phi_1'(\xi, \zeta) + \zeta^2 \Phi_1(\xi, \zeta) = -\frac{A}{2\pi} \delta(z - z_0) \tag{65}
\]

\[
\tilde{H}_1'(\xi, \zeta) + \zeta^2 \tilde{H}_1(\xi, \zeta) = -\frac{B_0}{4\pi} \delta(z - z_0) \tag{66}
\]

Here, \( \Phi_1(\xi, \eta) \) and \( \tilde{H}_1(\xi, \eta) \) are the depth dependent function. The solution of Eq. (65) is obtained using Fourier transform in \( z \) direction, i.e.,

\[
-\zeta^2 \Phi_1(\xi, \zeta) + \zeta^3 \Phi_1(\xi, \zeta) = -\frac{A e^{-i\zeta z_0}}{2\pi} \tag{67}
\]

Here, \( A e^{-i\zeta z_0} = \int_{-\infty}^{\infty} A(z) \delta(z - z_0) e^{-i\zeta z} dz \)

\[
\tilde{\Phi}(\xi, \eta) = \frac{A}{2\pi} \frac{e^{-i\zeta z_0}}{\xi^2 - \zeta^2} \tag{68}
\]

The solution of Eq. (66) is obtained in a similar way

\[
-\zeta^2 \tilde{H}_1(\xi, \zeta) + \zeta^3 \tilde{H}_1(\xi, \zeta) = -\frac{B_0}{4\pi} \xi \tag{69}
\]

\[
\tilde{H}_1(\xi, \zeta) = \frac{B_0}{4\pi} \frac{e^{-i\zeta z_0}}{\xi^2 - \zeta^2} + \frac{B_0}{8\pi} \tag{70}
\]

In order to get the solution in \( z \) domain, taking inverse Fourier transform of Eqs. (68) and (69), i.e.,

\[
\Phi_1(\xi, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A}{2\pi} \frac{e^{-i\zeta z_0} e^{i\zeta z}}{\xi^2 - \zeta^2} d\zeta \tag{71}
\]

\[
\tilde{H}_1(\xi, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{B_0}{4\pi} \frac{e^{-i\zeta z_0} e^{i\zeta z}}{\xi^2 - \zeta^2} d\zeta \tag{72}
\]

Upon rearrangement

\[
\Phi_1(\xi, z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \frac{A}{\xi^2 - \zeta^2} e^{i(z - z_0)} d\zeta \tag{73}
\]

\[
\tilde{H}_1(\xi, z) = \frac{B_0}{8\pi^2} \int_{-\infty}^{\infty} \frac{1}{\xi^2 - \zeta^2} e^{i(z - z_0)} d\zeta \tag{74}
\]

The evaluation of the integral in Eqs. (72) and (73) can be done by the residue theorem using a close contour (Watanabe, 2014; Haider et al. 2017; Remmert, 2012; Cohen, 2010; Krantz, 2007 [Appendix B]). Integration formula for Eqs. (72) and (73) are

\[
\Phi_1(\xi, z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \frac{1}{\xi^2 - \zeta^2} e^{i(z - z_0)} d\zeta = -\frac{A}{4\pi \xi \zeta} \sin \zeta z - z_0 \tag{75}
\]

\[
\tilde{H}_1(\xi, z) = \frac{B_0}{8\pi^2} \int_{-\infty}^{\infty} \frac{1}{\xi^2 - \zeta^2} e^{i(z - z_0)} d\zeta = -\frac{B_0}{8\pi \xi \zeta} \sin \zeta z - z_0 \tag{76}
\]

Eqs. (74), (75) represent the particular solution. The total solution is superposition of the complimentary solution Eqs. (63), (64) and particular solution Eqs. (74), (75), i.e.,

\[
\Phi_1(\xi, z) = C_1 \sin \zeta z + C_2 \cos \zeta z - \frac{A}{4\pi \xi \zeta} \sin \zeta z - z_0 \tag{77}
\]

\[
\tilde{H}_1(\xi, z) = D_1 \sin \zeta z + D_2 \cos \zeta z - \frac{B_0}{8\pi \xi \zeta} \sin \zeta z - z_0 \tag{78}
\]

The coefficients \( C_1, C_2, D_1, D_2 \) from Eqs. (76) and (77) are to be found from the Hankel transform of the boundary conditions Eq. (36), i.e.,

\[
(\tilde{\sigma}_{zz})_{\eta=\xi=0} = 0; \quad (\tilde{\sigma}_{zz})_{\xi=\eta=0} = 0 \tag{79}
\]

Here \( (\tilde{\sigma}_{zz})_{\eta=\xi=0} \) is the Hankel transform of order 0 for \( \tilde{\sigma}_{zz} \) and \( (\tilde{\sigma}_{zz})_{\xi=\eta=0} \) is the Hankel transform of order 1 for \( \tilde{\sigma}_{zz} \).

It should be noted that the total field Eqs. (76) and (77) must be used when satisfying the boundary conditions.

The differentiation properties of Eqs. (76) and (77) are,

\[
\frac{\partial (\tilde{\Phi}_b(\xi, \eta))}{\partial z} = -\tilde{\zeta} C_1 \cos \zeta z - \tilde{\zeta} C_2 \sin \zeta z - \frac{\tilde{\xi} B_0}{8\pi \eta_3} \cos \zeta |z - z_0| \tag{80}
\]

\[
\frac{\partial^2 (\tilde{\Phi}_b(\xi, \eta))}{\partial z^2} = -\tilde{\zeta} C_1 \sin \zeta z - \tilde{\zeta} C_2 \cos \zeta z + \frac{\tilde{\xi} B_0}{4\pi \xi \zeta} \sin \zeta |z - z_0| + \frac{B_0}{8\pi \eta_3} \cos \zeta |z - z_0| \tag{81}
\]

\[
\frac{\partial^2 (\tilde{H}_b(\xi, \eta))}{\partial z^2} = -\tilde{\zeta} D_1 \sin \zeta z - \tilde{\zeta} D_2 \cos \zeta z + \frac{\tilde{\xi} B_0}{8\pi \eta_3} \sin \zeta |z - z_0| \tag{82}
\]

Axis-symmetric stress equations in cylindrical coordinates are

\[
\sigma_{zz} = \lambda \text{div} \; \mathbf{u} + 2\mu \frac{\partial u_r}{\partial r} \tag{83}
\]

\[
\sigma_r = \mu \left( \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right) \tag{84}
\]

Here,

\[
\text{div}\mathbf{u} = \text{div} (\text{grad} \; \Phi + \text{curl} \; \tilde{H}) = \text{div} \; \text{grad} \; \Phi \tag{85}
\]

Using Eqs. (17), (34) and (85), into Eqs. (83) and (84) yield

\[
\sigma_{zz} = \lambda \left( 1 + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) + \frac{\partial^2 \Phi}{\partial z^2} \right) + 2\mu \frac{\partial^2 \Phi}{\partial z^2} + 2\mu \frac{1}{r} \frac{\partial (r \Phi)}{\partial r} \tag{86}
\]

\[
\sigma_r = \mu \left( 2 \frac{\partial^2 \Phi}{\partial r \partial z} + \frac{\partial}{\partial r} \left( 1 \frac{\partial (r \Phi)}{\partial r} \right) - \frac{\partial^2 H_0}{\partial z^2} \right) \tag{87}
\]

Applying Hankel transform of order 0 to \( \sigma_{zz} \), Eq. (86), yields \( (\tilde{\sigma}_{zz})_{\eta=\xi=0} \), i.e.,

\[
(\tilde{\sigma}_{zz})_{\eta=\xi=0} = \lambda \int_0^\infty \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial \Phi}{\partial r}) r\rho_0(\rho)(\rho)(\rho) \tag{88}
\]

Calculating the individual terms of Eq. (88) using ref. (Giurgiu, 2014 pp. 611–612)

\[
\int_0^\infty \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial \Phi}{\partial r}) r\rho_0(\rho)(\rho) \tag{89}
\]

\[
\int_0^\infty \frac{\partial^2 \Phi}{\partial z^2} r\rho_0(\rho) \tag{90}
\]

\[
\int_0^\infty \frac{1}{r} \frac{\partial^2 (r \Phi)}{\partial z^2} r\rho_0(\rho) \tag{91}
\]
Substituting of Eqs. (89), (90) and (91) into Eq. (88) yields

\[
\begin{align*}
(\sigma_{rz})_h &= -\lambda \xi^2 \Phi_0 + (\lambda + 2\mu) \frac{\partial^2 \Phi_0}{\partial z^2} + 2\mu \xi \frac{\partial \tilde{H}_0}{\partial z} + 2\mu \xi^2 \frac{\partial^2 \tilde{H}_0}{\partial z^2} \\
(\sigma_{rz})_h &= -2\mu \xi \frac{\partial \Phi_0}{\partial z} - \mu \xi^2 \frac{\partial \tilde{H}_0}{\partial z} - \mu \frac{\partial^2 \tilde{H}_0}{\partial z^2} 
\end{align*}
\]  

(92)

Applying Hankel transform of order 1 to \(\sigma_{rz}\), Eq. (87), yields

\[
\sigma_{rz} = \mu \left( 2 \frac{\partial^2 \Phi_0}{\partial r \partial z} + \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial (r H_0)}{\partial r} \right) - \frac{\partial^2 H_0}{\partial z^2} \right)
\]  

(93)

Calculating the individual terms of Eq. (93) using from ref. (Giurgiutiu, 2014; pp. 611–612)

\[
(\sigma_{rz})_h = \mu \left( 2 \int_0^\infty \frac{\partial^2 \Phi_0}{\partial r \partial z} r J_1(\xi r) dr + \int_0^\infty \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial (r H_0)}{\partial r} \right) r J_1(\xi r) dr - \int_0^\infty \frac{\partial^2 H_0}{\partial z^2} r J_1(\xi r) dr \right)
\]  

(94)

\[
\int_0^\infty \frac{\partial^2 \Phi_0}{\partial r \partial z} r J_1(\xi r) dr = -\xi \frac{\partial \Phi_0}{\partial z}
\]  

(95)

\[
\int_0^\infty \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial (r H_0)}{\partial r} \right) r J_1(\xi r) dr = -\xi^2 \tilde{H}_0
\]  

(96)

\[
\int_0^\infty \frac{\partial^2 H_0}{\partial z^2} r J_1(\xi r) dr = -\frac{\partial^2 \tilde{H}_0}{\partial z^2}
\]  

(97)

Substituting of Eqs. (95)-(97) into Eq. (94) yields

\[
(\sigma_{rz})_h = -2\mu \xi \left( \frac{\partial \Phi_0}{\partial z} - \mu \xi^2 \frac{\partial \tilde{H}_0}{\partial z} - \mu \frac{\partial^2 \tilde{H}_0}{\partial z^2} \right)
\]  

(98)

Using Eqs. (76), (80) and (81) into Eq. (92) gives,

\[
(\sigma_{rz})_h = -\lambda \xi^2 \left( C_1 \sin \xi \rho z + C_2 \cos \xi \rho z - \frac{A}{4\pi \xi \rho} \sin \xi \rho |z - z_0| \right) - \lambda \xi^2 (\lambda + 2\mu) \left( C_1 \sin \xi \rho z + C_2 \cos \xi \rho z - \frac{A}{4\pi \xi \rho} \sin \xi \rho |z - z_0| \right) + 2\mu \xi \left( \xi D_1 \cos \xi \rho z D_2 \sin \xi \rho z - \frac{\xi B_0}{8\pi \xi} \cos \xi \rho |z - z_0| \right)
\]  

(99)

Upon rearrangement,

\[
\begin{align*}
(\sigma_{rz})_h &= -\lambda \xi^2 \left( \lambda + 2\mu \right) \left( C_1 \sin \xi \rho z + C_2 \cos \xi \rho z - \frac{A}{4\pi \xi \rho} \sin \xi \rho |z - z_0| \right) + 2\mu \xi \left( \xi D_1 \cos \xi \rho z D_2 \sin \xi \rho z - \frac{\xi B_0}{8\pi \xi} \cos \xi \rho |z - z_0| \right)
\end{align*}
\]  

(100)

Here,

\[
\{ \lambda \xi^2 + \lambda \xi^2 (\lambda + 2\mu) \} = -\mu (\xi^2 - \xi^2)
\]  

(101)

Hence, Eq. (100) becomes

\[
\begin{align*}
(\sigma_{rz})_h &= \mu \left( \xi^2 - \xi^2 \right) \left( C_1 \sin \xi \rho z + C_2 \cos \xi \rho z - \frac{A}{4\pi \xi \rho} \sin \xi \rho |z - z_0| \right) + 2\mu \xi \left( \xi D_1 \cos \xi \rho z D_2 \sin \xi \rho z - \frac{\xi B_0}{8\pi \xi} \cos \xi \rho |z - z_0| \right)
\end{align*}
\]  

(102)
Eqs. (106)-(109) are a set of four equations with four unknowns. The equations can be separated into a couple of two equations with two unknowns, one for symmetric motion and one for anti-symmetric motion.

3.1. Symmetric Lamb wave solution

Addition of the Eqs. (106) and (107), and subtraction of the Eq. (109) from Eq. (108), yield

\[
\begin{bmatrix}
\left(\xi^2 - \eta^2\right) \cos \xi d \\
-2\xi \eta \cos \xi d \\
\end{bmatrix}
\begin{bmatrix}
C_2 \\
D_1 \\
\end{bmatrix}
= \begin{bmatrix}
\left(\xi^2 - \eta^2\right) A \frac{\partial}{\partial \theta} \sin \xi |d - z_0| \\
0 \\
\end{bmatrix}
\]

Eq. (110) represents an algebraic system that can be solved for $C_2, D_1$. The system determinant does not vanish

\[
\begin{bmatrix}
C_2 \\
D_1 \\
\end{bmatrix}
= \begin{bmatrix}
\left(\xi^2 - \eta^2\right) \cos \xi d \\
2\xi \eta \cos \xi d \\
\end{bmatrix}
= 0
\]

Let,

\[
P_3 = \left(\xi^2 - \eta^2\right) \cos \xi |d - z_0| + \left(\xi^2 \eta^2 - \eta^2 \right) \cos \xi |d - z_0|
\]

$P_3$ is the source term for symmetric solution which contains source potentials $A$ and $B_0$.

Upon substitution of $d_1 = d - z_0$ into Eq. (111)

\[
P_3 = \left(\xi^2 - \eta^2\right) \cos \xi |d_1| + \left(\xi^2 \eta^2 - \eta^2 \right) \cos \xi |d_1|
\]

Here $d_1$ is the source depth (Fig. 2). Then,

\[
\begin{bmatrix}
C_2 \\
D_1 \\
\end{bmatrix}
= \begin{bmatrix}
\left(\xi^2 - \eta^2\right) \cos \xi d \\
2\xi \eta \cos \xi d \\
\end{bmatrix}
\begin{bmatrix}
P_3 \\
0 \\
\end{bmatrix}
\]

Upon rearrangement

\[
\begin{bmatrix}
C_2 \\
D_1 \\
\end{bmatrix}
= \frac{1}{D_1} \begin{bmatrix}
P_3 \left(\xi^2 - \eta^2\right) \cos \xi d \\
2\xi \eta \cos \xi d \\
\end{bmatrix}
\]

Here,

\[
D_1 = \left(\xi^2 - \eta^2\right) \cos \xi d \\
-2\xi \eta \cos \xi d \\
\end{bmatrix}
= \left(\xi^2 - \eta^2\right)^2 \cos \xi d \sin \xi \sin d + 4\xi^2 \eta^2 / \xi^2 \cos \xi \sin \xi \sin d
\]

By equating to zero the $D_1(\xi)$ term of Eq. (115), one gets the symmetric Rayleigh-Lamb wave equation.

3.2. Anti-symmetric Lamb wave solution

Subtraction of the Eq. (106) from (107), and addition of the Eqs. (108) and (109) yield

\[
\begin{bmatrix}
\left(\xi^2 - \eta^2\right) \sin \xi d \\
2\xi \eta \sin \xi d \\
\end{bmatrix}
\begin{bmatrix}
C_1 \\
D_2 \\
\end{bmatrix}
= \begin{bmatrix}
\xi A / 2\pi \cos \xi |d - z_0| \\
\xi \eta / 2\pi \cos \xi |d - z_0| \\
\end{bmatrix}
\]

Eq. (116) represents an algebraic system that can be solved for $C_1, D_2$. The system determinant does not vanish

\[
D_A = \begin{bmatrix}
\left(\xi^2 - \eta^2\right) \sin \xi d \\
2\xi \eta \cos \xi d \\
\end{bmatrix}
= 0
\]

Let,

\[
P_A = \left(\xi A / 2\pi \cos \xi |d - z_0| + \xi \left(\xi^2 - \eta^2\right) B_0 / \eta \xi^2 \right) \sin \xi |d - z_0|
\]

$P_A$ is the source term for anti-symmetric solution which contains source potentials $A$ and $B_0$.

Upon substitution of $d_1 = d - z_0$ into Eq. (117)

\[
P_A = \left(\xi A / 2\pi \cos \xi d_1 + \xi \left(\xi^2 - \eta^2\right) B_0 / \eta \xi^2 \right) \sin \xi d_1
\]

Then,

\[
\begin{bmatrix}
C_1 \\
D_2 \\
\end{bmatrix}
= \begin{bmatrix}
\left(\xi^2 - \eta^2\right) \sin \xi d \\
2\xi \eta \sin \xi d \\
\end{bmatrix}
\begin{bmatrix}
P_A \\
0 \\
\end{bmatrix}
\]

Upon rearrangement

\[
\begin{bmatrix}
C_1 \\
D_2 \\
\end{bmatrix}
= \frac{1}{D_A} \begin{bmatrix}
P_A \left(\xi^2 - \eta^2\right) \sin \xi d \\
2\xi \eta \sin \xi d \\
\end{bmatrix}
\]

Here,

\[
D_A = \left(\xi^2 - \eta^2\right) \sin \xi d \\
-2\xi \eta \sin \xi d \\
\end{bmatrix}
= \left(\xi^2 - \eta^2\right)^2 \sin \xi d \sin \xi \sin d + 4\xi^2 \eta^2 / \xi^2 \sin \xi \sin d
\]

By equating to zero the $D_1(\xi)$ term of Eq. (121), one gets the anti-symmetric Rayleigh-Lamb wave equation.

3.3. Complete solution in the wavenumber domain

Eq. (13) gives the displacement $u_2$, in terms of potentials, i.e.,

\[
u_2 = \frac{\partial \Phi}{\partial \xi} + \frac{1}{r} \frac{\partial (r H_0)}{\partial r}
\]

After applying Hankel transform of zero order to Eq. (122), displacement $u_2$ becomes,

\[
u_2 = \frac{\partial \Phi}{\partial \xi} + \xi \bar{H}_0
\]

By substituting of Eqs. (76) and (77) into Eq. (123) yields the expressions for out-of-plane velocity in the wavenumber domain in terms of coefficients $C_1, C_2, D_1, D_2$, which are functions of the wavenumber $\xi$

\[
u_2 = \left[C_1 \cos \xi d - C_2 \sin \xi d - \left(\frac{A}{4\pi}\right) \cos \xi |d - z_0|\right]
\]

Evaluation of Eq. (124) at the plate top surface $\xi_{1z = d}$:

\[
u_2 = \left[C_1 \cos \xi d - C_2 \sin \xi d - \left(\frac{A}{4\pi}\right) \cos \xi |d - z_0|\right]
\]

The complete solution is the superposition of the symmetric and antisymmetric solutions. By using Eqs. (144) and (120)
into Eq. (125) and obtain,

\[
\tilde{u}_z = \frac{1}{\xi} \left( P \frac{N_0}{D_0} + P A \frac{N_A}{D_A} - \xi \left( A \frac{4\pi}{\sqrt{\xi}} \right) \cos \xi d - z_0 \right) - \xi^3 B_0 \frac{\sin \xi d - z_0}{8\pi \xi} \right)
\]

(126)

Here
\[ N_s = 2\xi^2 \xi \sin \xi d - \xi (\xi^2 - \xi^2_1) \xi \sin \xi d \]
\[ N_A = 2\xi^2 \xi^2_0 \cos \xi d + \xi^3 (\xi^2 - \xi^2_0) \sin \xi d \cos \xi d \]
Upon substitution of \( d_1 = d - z_0 \) into Eq. (127)
\[
\tilde{u}_z = \frac{1}{\xi} \left( P \frac{N_0}{D_0} + P A \frac{N_A}{D_A} - \xi \left( A \frac{4\pi}{\sqrt{\xi}} \right) \cos \xi d_1 - \xi^3 B_0 \frac{\sin \xi d_1}{8\pi \xi} \right)
\]

(128)

3.4. Complete solution in the physical domain

Lamb wave solution in the physical domain by applying inverse Hankel transforms to Eq. (128) and obtain,
\[
u_z = \int_0^\infty \frac{1}{\xi} \left( P \frac{N_0}{D_0} + P A \frac{N_A}{D_A} - \xi \left( A \frac{4\pi}{\sqrt{\xi}} \right) \cos \xi \xi d_1 \right) j_0(\xi \xi d) d\xi
\]

(129)

Upon rearrangement of Eq. (129)
\[
u_z = \int_0^\infty \left( P \frac{N_0}{D_0} + P A \frac{N_A}{D_A} - \xi \left( A \frac{4\pi}{\sqrt{\xi}} \right) \cos \xi \xi d_1 \right) j_0(\xi \xi) d\xi
\]

(130)

Let,
\[
u_z \equiv \nu_{z1} + \nu_{z2} + \nu_{z3}
\]

(131)

where,
\[
u_{z1} = \int_0^\infty \left( P \frac{N_0}{D_0} + P A \frac{N_A}{D_A} \right) j_0(\xi \xi) d\xi
\]
\[
u_{z2} = \int_0^\infty \left( A \frac{4\pi}{\sqrt{\xi}} \right) \cos \xi \xi d_1 \right) j_0(\xi \xi) d\xi
\]
\[
u_{z3} = \int_0^\infty \left( \frac{1}{\sqrt{\xi}} \frac{B_0}{8\pi \xi} \sin \xi \xi d_1 \right) j_0(\xi \xi) d\xi
\]

(132)

(133)

(134)

The integration of Eqs. (132)–(134) are presented in Appendix C. After integrating Eqs. (132)–(134), the displacement \( u_z \) becomes
\[
u_z = \pi i \left( \sum_{j=0}^{n} \left[ \frac{P}{D_0} \frac{N_0}{D_0} \frac{N_A}{N_A} \frac{N_A}{D_A} \right] \right) \left[ \frac{1}{\xi} \left( \frac{A}{4\pi} \right) \cos \xi \xi d_1 \right] j_0(\xi \xi) d\xi
\]
\[
u_z = \pi i \left( \sum_{j=0}^{n} \left[ \frac{P}{D_0} \frac{N_0}{D_0} \frac{N_A}{D_A} \right] \right) \left[ \frac{1}{\xi} \left( \frac{A}{4\pi} \right) \cos \xi \xi d_1 \right] j_0(\xi \xi) d\xi
\]
\[
u_z = \pi i \left( \sum_{j=0}^{n} \left[ \frac{P}{D_0} \frac{N_0}{D_0} \frac{N_A}{D_A} \right] \right) \left[ \frac{1}{\xi} \left( \frac{A}{4\pi} \right) \cos \xi \xi d_1 \right] j_0(\xi \xi) d\xi
\]
\[
u_z = \pi i \left( \sum_{j=0}^{n} \left[ \frac{P}{D_0} \frac{N_0}{D_0} \frac{N_A}{D_A} \right] \right) \left[ \frac{1}{\xi} \left( \frac{A}{4\pi} \right) \cos \xi \xi d_1 \right] j_0(\xi \xi) d\xi
\]

(135)

After rearranging Eq. (135) and reinstating explicitly the harmonic time dependency, we get,
\[
u_z = \pi i \left( \sum_{j=0}^{n} \left[ \frac{P}{D_0} \frac{N_0}{D_0} \frac{N_A}{D_A} \right] \right) \left[ \frac{1}{\xi} \left( \frac{A}{4\pi} \right) \cos \xi \xi d_1 \right] j_0(\xi \xi) d\xi
\]
\[
u_z = \pi i \left( \sum_{j=0}^{n} \left[ \frac{P}{D_0} \frac{N_0}{D_0} \frac{N_A}{D_A} \right] \right) \left[ \frac{1}{\xi} \left( \frac{A}{4\pi} \right) \cos \xi \xi d_1 \right] j_0(\xi \xi) d\xi
\]
\[
u_z = \pi i \left( \sum_{j=0}^{n} \left[ \frac{P}{D_0} \frac{N_0}{D_0} \frac{N_A}{D_A} \right] \right) \left[ \frac{1}{\xi} \left( \frac{A}{4\pi} \right) \cos \xi \xi d_1 \right] j_0(\xi \xi) d\xi
\]
\[
u_z = \pi i \left( \sum_{j=0}^{n} \left[ \frac{P}{D_0} \frac{N_0}{D_0} \frac{N_A}{D_A} \right] \right) \left[ \frac{1}{\xi} \left( \frac{A}{4\pi} \right) \cos \xi \xi d_1 \right] j_0(\xi \xi) d\xi
\]

(136)

Assume that,
\[
u_z = \nu_{z1} + \nu_{z2} + \nu_{z3}
\]

(137)

where,
\[
u_{z1} = \pi i \left( \sum_{j=0}^{n} \left[ \frac{P}{D_0} \frac{N_0}{D_0} \frac{N_A}{D_A} \right] \right) \left[ \frac{1}{\xi} \left( \frac{A}{4\pi} \right) \cos \xi \xi d_1 \right] j_0(\xi \xi) d\xi
\]
\[
u_{z2} = \pi i \left( \sum_{j=0}^{n} \left[ \frac{P}{D_0} \frac{N_0}{D_0} \frac{N_A}{D_A} \right] \right) \left[ \frac{1}{\xi} \left( \frac{A}{4\pi} \right) \cos \xi \xi d_1 \right] j_0(\xi \xi) d\xi
\]
\[
u_{z3} = \pi i \left( \sum_{j=0}^{n} \left[ \frac{P}{D_0} \frac{N_0}{D_0} \frac{N_A}{D_A} \right] \right) \left[ \frac{1}{\xi} \left( \frac{A}{4\pi} \right) \cos \xi \xi d_1 \right] j_0(\xi \xi) d\xi
\]

(138)

(139)

(140)

\( u_{z1} \) is the out-of-plane displacement containing Lamb wave mode and \( u_{z2}, u_{z3} \) are the out-of-plane displacements of bulk wave due to excitation potentials \( A \) and \( B \) respectively. Bulk waves are categorized into longitudinal (pressure) and transverse (shear) waves. The displacements \( u_{z1} \) and \( u_{z3} \) are the longitudinal and transverse bulk waves respectively. Longitudinal and transverse bulk wave is generated due to pressure excitation and shear excitation potentials respectively.

Our theoretical derivation considers that both bulk wave and Lamb waves are excited by the AE event. The Lamb waves consist of patterns of thickness direction standing waves (the Lamb wave modes) that travel radially as propagating waves and eventually reach the AE sensor. Whereas, bulk waves (pressure and shear) propagate spherically from the source. In a plate, the bulk waves
that propagate at nonzero angles with respect to the plate midplane would bounce from the top and bottom plate surfaces, interact with each other, and eventually settle down as Lamb wave modes. Whereas the bulk waves that propagate at zero angle along the plate mid surface would not interfere with the plate surfaces and would travel directly to the sensor location. The theoretical formulations derived in this paper shows that the Lamb waves and the bulk waves have different expressions Eqs. (138)-(140). Therefore, bulk wave profiles are not related to the symmetric or asymmetric content of the signal.

3.5. AE elastic wave propagation

AE elastic waves will be generated by the AE event. AE elastic waves will propagate through the structure according to structural transfer function. The out-of-plane displacement of the elastic waves can be captured by conventional AE transducer installed on the surface of the structure as shown in Fig. 5.

For numerical analysis, 304-stainless steel material with 6 mm thickness was chosen as a case study. The signal was received at 500 mm distance from the source. The excitation source was located at mid-plane. A very short peak time of 3 μs was used for case study. Later, the effects of source of depth and propagation distance were included into the study. The details methodology of numerical analysis is described in Fig. 6.

4. Numerical studies

This section includes elastic wave simulation in a plate due to excitation potentials.

4.1. Time dependent excitation potentials

The time-dependent excitation potentials depend on time-dependent energy released from a crack. A real AE source releases energy during a finite time period. At the beginning, the rate of energy released from a crack increases sharply with time and reaches a maximum peak value within very short time; then decreases asymptotically toward the steady-state value, usually zero. Time profile of AE source may follow cosine bell function (Hamstad, 2010) or error function. In this research, a Gaussian pulse is used to model the growth of the excitation potentials during the AE event; as a result, the actual excitation potential follows the error function variation in the time domain. Cuadra et al. (2016) obtained similar time profile of AE energy from a crack by using 3D computation method. Figs. 7a and 8a show the time rate of pressure and shear potentials. The corresponding equations are,

\[
\frac{\partial A^*}{\partial t} = A_{0} t^2 e^{-\frac{t^2}{\tau}}
\]

\[
\frac{\partial B_0^*}{\partial t} = B_{00} t^2 e^{-\frac{t^2}{\tau}}
\]  

(141)

Here, \( A_0 \) and \( B_{00} \) are the scaling factors and \( \tau \) is the peak time of the pulse. Time profile of the potentials are to be evaluated by integrating Eq. (141), i.e.,

\[
A^* = A_0 \left( \sqrt{\pi} \operatorname{erf}(t/\tau) - 2t e^{-\frac{t^2}{\tau}} \right)
\]

\[
B_0^* = B_{00} \left( \sqrt{\pi} \operatorname{erf}(t/\tau) - 2t e^{-\frac{t^2}{\tau}} \right)
\]

(142)

Time profile of potentials are shown in Figs. 7b and 8b. The key characteristics of the excitation potentials are:

- peak time: time required to reach the time rate of potential to maximum value
- rise time: time required to reach the potential to 98% of maximum value or steady state value
- peak value: maximum value of time rate of potential
- maximum potential: maximum value of time profile of potential

Amplitude of the source \( A \) and \( B_z \) for unit volume is,

\[
A = \frac{1}{c_p^2} A^*
\]

\[
B_0 = \frac{d}{c_t} B_{00}^*
\]  

(143)

Here, \( d \) is the plate half thickness.
4.2. Phase velocity and group velocity dispersion curves

Solution of the Rayleigh–Lamb equation for symmetric and antisymmetric modes Eqs. (115) and (121) yields the wave number and hence the phase velocity for each given frequency. Multiple solutions exist, hence multiple Lamb modes also exist. Differentiation of phase velocity with respect to frequency yields the group velocity. The phase velocity dispersion curves and group velocity dispersion curves are shown in Fig. 9a and b respectively. The existence of certain Lamb mode depends on the plate thickness and frequency. The fundamental S0 and A0 modes will always exist. The phase velocity (Fig. 9a) is associated with the phase difference between the vibrations observed at two different points during the passage of the wave. The phase velocity is used to calculate the wavelength of each mode.

The fundamental wave mode of symmetric (S0) and antisymmetric (A0) is considered in this analysis. Signal is received at some distance from the point of excitation potentials, only the modes with real-valued wavenumbers are included in the simulation. The corresponding modes for imaginary and complex-valued wavenumbers are ignored because they propagate with decaying amplitude; at sufficiently far from the excitation point their amplitudes are negligible. Since different Lamb wave modes traveling with different wave speeds exist simultaneously, the excitation potentials will generate S0 and A0 wave packets. The group velocity (Fig. 9b) of Lamb waves is important when examining the traveling of Lamb wave packets. These wave packets will travel independently through the plate and will arrive at different times. Due to the multi-mode character of elastic Lamb wave propagation, the received signal has at least two separate wave packets, S0 and A0. All wave modes propagate independently in the structure. The final waveform will be the superposition of all the propagating waves and will have the contribution from each Lamb wave mode.

4.3. Example of AE elastic wave propagation in a 6 mm plate

A test case example is presented in this section to show how AE elastic waves propagate over a certain distance using exса-
tion potentials. A 304-stainless steel plate with 6 mm thickness was chosen for this purpose. The signal was received at 500 mm distance from the source. Excitation sources were located at midplane of the plate ($d_{i} = 3$ mm). The time profile of excitation potentials (Figs. 7b and 8b) were used to simulate AE waves in the plate. Excitation potential can be calculated from the time rate of potential released during crack propagation. Peak time, rise time, peak value and maximum potential of excitation potentials are $3 \mu s$, $6.6 \mu s$, 0.28 $\mu W$/kg and $1 \mu J$/kg respectively. Based on this information a numerical study on AE Lamb wave propagation is conducted. Several researchers investigate the potential effect of rise time on AE signal by varying the source rise time in-between 0.1 $\mu s$ to 15 $\mu s$ [Hamstad, 2010; Sause et al., 2013, Michaels et al., 1981].

In this article, each excitation potential was considered separately to simulate the Lamb waves. Total released energy from a crack can be decomposed to pressure and shear excitation potentials. This article presents the individual effect of the unit amplitude of potentials (1 $\mu J$/kg) on AE signals. However, distributing the energy between the pressure and shear potentials for real AE source is very important and we recommend it for future study for specific damage assumptions (e.g., a crack propagating under Mode I fracture may have a different mix of shear/potential potentials than a crack propagating under Mode II or Mode III fracture modes; similarly for mixed-mode fracture). Broadband AE transducers can detect the out-of-plane displacement components of the resulting AE signals. Therefore, the aim of this numerical simulation is to simulate the out-of-plane displacement that would be recorded by an AE sensor placed on the plate surface at some distance away from the source. However, the resulting AE signal is the superposition of several Lamb waves modes and bulk waves existing for the particular frequency-thickness combination under consideration.

In this numerical analysis, we considered the different modes separately to understand the impact of excitation potentials. Total released energy from a crack can be decomposed to pressure and shear excitation potentials. Fig. 10a shows the Lamb wave ($S_{0}$ and $A_{0}$ mode) and bulk wave propagation using pressure excitation potential. Signals are normalized by their individual peak amplitudes, i.e., amplitude/peak amplitude. It can be inferred from the figures that both $S_{0}$ and $A_{0}$ are dispersive. $S_{0}$ contains high frequency component at which it is dispersive, whereas, $A_{0}$ contains low frequency component at which it is also dispersive. Bulk wave shows nondispersive behavior, but the peak amplitude is not significant compared to peak $S_{0}$ and $A_{0}$ amplitude. The notable characteristic is that, peak amplitude of $A_{0}$ is higher than the peak $S_{0}$ amplitude. Fig. 10b shows the Lamb wave ($S_{0}$ and $A_{0}$ mode) and bulk wave propagation using shear potential only.

By comparing Figs. 10a and b the important observation can be made as, shear excitation potential has more contribution to the peak $S_{0}$ amplitude over pressure excitation potentials whereas, pressure and shear excitation potentials have almost equal contribution to the peak $A_{0}$ amplitude. However, their contribution to the $S_{0}$ and $A_{0}$ peak amplitude might change due to change in source depth and propagating distance. The proceeding section discusses the effect of source depth and propagating distance in AE elastic wave propagation using excitation potentials.

4.4. Effect of source depth

To study the effect of source depth, a 6 mm thick plate was chosen. The signal was received at 500 mm distance from the source. The excitation sources were located at the top surface, 1.5 mm and 3 mm deep (mid-plane) from the top surface. Figs. 11–13 show the out-of-plane displacement ($S_{0}$ and $A_{0}$ mode, bulk wave) vs time at 500 mm propagation distance in 6 mm thick plate for different source location. Fig. 11 shows that $A_{0}$ mode appears only when using pressure potential on the top surface whereas, $S_{0}$ mode and bulk wave appear while using shear potential. The displacements $u_{S_{0}}$ (Eq. (139)) and $u_{A_{0}}$(Eq. (140)) are the pressure and shear bulk waves generated due to pressure excitation and shear excitation potentials, respectively. Examination of these equations reveals that displacement $u_{S_{0}}$ due to pressure excitation potential has a multiplication factor $d_{1}$ whereas the displacement $u_{A_{0}}$ due to the shear potential does not have such a multiplication factor. Therefore, when $d_{1} = 0$, only the bulk wave $u_{A_{0}}$ due to pressure potential excitation becomes zero whereas $u_{S_{0}}$ due to shear potential remains nonzero.

Another notable characteristic is that pressure potential has the contribution to the trailing edge (low frequency component) of $A_{0}$ wave packet for the top-surface source. With increasing source depth the effect of high amplitude of low frequency decreases (Figs. 12a and 13a) in $A_{0}$ signal. By comparing Figs. 12, and 13, clearly amplitude scales show that the increase in peak $A_{0}$ amplitude and decrease in peak $S_{0}$ amplitude with increasing source depth while using pressure excitation potential only.

Figs. 11b–13b show some qualitative and quantitative changes in spectrum while using shear excitation potentials only. The peak amplitude of $S_{0}$ signal decreases and $A_{0}$ signal increases with increasing source depth while using shear excitation potential only (Figs. 12b and 13b). Qualitative change in $S_{0}$ signal refers to the change in the frequency content of the signal. It should be noted that the peak amplitude of $S_{0}$ due to shear excitation potential located at 1.5 mm depth from the top surface is more significant than pressure potential (Fig. 12).

For all AE source location, the shear potential part of the AE source has more contribution to the peak $S_{0}$ amplitude than pressure potential. The peak amplitude of bulk waves increases with increasing source depth (Figs. 12 and 13). However, the peak amplitude of bulk wave is much smaller than the peak $A_{0}$ and $S_{0}$ amplitudes. Therefore, bulk wave may not be significant in real AE signal.

4.5. Effect of peak time

This section discusses the effect of peak time on AE elastic waves due to excitation potentials. The numerical study was done for a selection of peak times of 3 $\mu s$ and 9 $\mu s$. For brevity, only the AE signal due to pressure potential is considered in this case. Fig. 14 shows (a) the time profile of pressure potential ($A_{0}^{*}$) and (b) frequency response of the pressure potential. Since AE signals have a wideband response; but a real AE sensor does not have ultimate high frequency response; therefore, the signals were filtered at 10 kHz and 800 kHz frequency. Fig. 14(b) shows that lower peak time signal has higher amplitude response than higher peak time signal. Higher peak time excitation signal (9 $\mu s$) is dominant at low frequency. Therefore, low frequency response may be observed in resulting AE signal.

Fig. 15 shows the out-of-plane displacement in time domains for propagating distance of 500 mm from the excitation source in a 6 mm plate. These figures are for the source located at mid plane of the plate. Clearly, the amplitude scales show that rise time has an effect on low frequency $S_{0}$ and $A_{0}$ amplitudes. The shapes in the time domain of the waveform changes significantly with changing peak time. By comparing Fig. 15(a) and (b) it can be inferred that $S_{0}$ has low frequency component at the beginning of the signal, whereas $A_{0}$ has low frequency components at the end of the signal for peak time of 9 $\mu s$. Peak time might have more influence on amplitude and shape if the source located near the top surface rather than at mid plane. A future study on the synergistic effect of rise time and source depth is recommended in the future study section.
Fig. 10. Lamb wave ($S_0$ and $A_0$ mode) and bulk wave propagation at 500 mm distance in 6 mm 304-steel plate for (a) pressure potential excitation (b) shear potential excitation (peak time = 3 μs) located at 3.0 mm depth from top surface.

Fig. 11. Lamb wave ($S_0$ and $A_0$ mode) and bulk wave propagation at 500 mm distance in 6 mm 304-steel plate for (a) pressure potential excitation (b) shear potential excitation (peak time = 3 μs) located on top surface.
Fig. 12. Lamb wave ($S_0$ and $A_0$ mode) and bulk wave propagation at 500 mm distance in 6 mm 304-steel plate for (a) pressure potential excitation (b) shear potential excitation (peak time = 3 μs) located at 1.5 mm depth from top surface.

Fig. 13. Lamb wave ($S_0$ and $A_0$ mode) and bulk wave propagation at 500 mm distance in 6 mm 304-steel plate for (a) pressure potential excitation (b) shear potential excitation (peak time = 3 μs) located at 3.0 mm depth from top surface.
The attenuation of the peak amplitude of the signal as a function of propagating distance was also determined. Figs. 16 and 17 show the normalized peak amplitude of out-of-displacement ($S_0$, $A_0$, bulk wave) against propagation distance from 100 mm to 500 mm for pressure potential, shear potential respectively. The amplitudes are normalized by their individual peak amplitudes. A 6 mm thick 304-steel plate with both excitation potentials are considered. Potentials are located at mid-plane with peak time of 3 $\mu$s.

Figs. 16 and 17 show significant attenuation of $S_0$, $A_0$ and bulk wave signal with propagating distance 100 mm to 50 mm. However, larger attenuation of peak bulk wave amplitude is observed compared to peak $S_0$ and $A_0$ amplitudes. Therefore, far from the source bulk wave becomes less significant and may not be captured by the AE transducer. Another important observation is that peak $S_0$ amplitude attenuates more than peak $A_0$ amplitude for both excitation potentials. The attenuation of the peak amplitude is expected due to the dispersion of the signal.

4.6. Effect of propagation distance

Fig. 14. (a) Time profile of pressure potential ($A^*$) (b) frequency response of the pressure potential.

Fig. 15. Lamb wave ($S_0$ and $A_0$ mode) propagation at 500 mm distance in 6 mm 304-steel plate for pressure potential excitation (a) peak time of 3 $\mu$s and (b) peak times of 9 $\mu$s.

Fig. 16. Effect of pressure excitation potential: variation of out-of-plane displacement ($S_0$, $A_0$ and bulk wave) with propagation distance in 6 mm 304-steel plate for source (peak time = 3 $\mu$s) located at the mid plane.
5. Summary, conclusions, and future work

5.1. Summary

Cylindrical coordinate solutions of axis symmetric circular crest ed elastic waves that appear due to sudden energy release were analyzed through a Helmholtz potential approach. The inhomogeneous elastodynamic Navier–Lame equations were expressed as a system of wave equations in terms of unknown scalar and vector solution potentials, $\Phi$, $H$, and known scalar and vector excitation potentials, $A^e$, $B^e$. The excitation potentials $A^e$, $B^e$ were traced to the energy released during an incremental crack propagation.

The solution was readily obtained through direct and inverse integral transforms and application of the residue theorem. The resulting solution took the form of a series expansion containing the superposition of all the Lamb waves modes existing for the particular frequency-thickness combination under consideration. Bulk wave solution is also recovered due to the effect of the excitation potentials.

A numerical study of the AE elastic wave propagation in a 6 mm thick 304-steel plate was conducted in order to predict the out-of-plane displacement that would be recorded by an AE sensor placed on the plate surface at some distance away from the source. Parameter studies were performed to evaluate: (a) the effect of the pressure and shear potentials; (b) the effect of the thickness-wise location of the excitation potential sources varying from mid-plane to the top surface (source depth effect); (c) the effect of peak time (d) the effect of propagating distance away from the source.

To understand the signal attenuation with propagating distance, a numerical simulation was also conducted for propagating distance of 100 mm. Fig. 18 shows the AE elastic wave propagation at 100 mm distance from the source. The effect of the propagating distance was found to produce increasingly larger losses in $S_0$, $A_0$ and bulk peak signal amplitudes as the propagation distance varied between 100 mm and 500 mm (Figs. 13 and 18). Since the medium was assumed lossless, this attenuation is propagation distance is attributable to signal dispersion, which is to be expected due to the wideband character of the AE excitation.

Fig. 17. Effect of shear excitation potential; variation of out-of-plane displacement ($S_0$, $A_0$ and bulk wave) with propagation distance in a 6-mm 304-steel plate for source (peak time = 3 $\mu$s) located at the mid plane.

Fig. 18. Lamb wave ($S_0$ and $A_0$ mode) and bulk wave propagation at 100 mm distance in 6 mm 304-steel plate for (a) pressure potential excitation (b) shear potential excitation (peak time = 3 $\mu$s) located at 3.0 mm depth from top surface.
5.2. Conclusions

This article has shown that pressure and shear source potentials can be used to model the elastic wave generation and propagation due to an AE event associated with incremental crack growth. The advantage of this approach is to decouple the inhomogeneous Navier–Lame equations.

The numerical studies performed over a range of parameters have shown that:

i. Both \( A_0 \) and \( S_0 \) modes generated by an AE excitation in a 6 mm steel plate are dispersive. The high frequency components of the AE excitation are responsible for the dispersive part of the \( S_0 \) mode, whereas the low frequency components of the AE excitation are responsible for the dispersive part of the \( A_0 \) mode.

ii. The peak amplitude of \( A_0 \) mode is higher than the peak amplitude of the \( S_0 \) mode for all cases.

iii. The amplitude of bulk wave is much smaller than peak \( A_0 \) and \( S_0 \) amplitude. Therefore, peak amplitude of bulk waves may not be significant in real AE signal.

iv. For mid-plane AE source location, the shear potential part of the AE source has more contribution to the peak \( S_0 \) amplitude whereas the pressure and shear potentials have the almost equal contribution to the peak \( A_0 \) amplitude.

v. An increase in the source depth increases the peak \( A_0 \) and \( S_0 \) amplitude while using pressure potential only, whereas, an increase in the source depth increases the peak \( A_0 \) and decreases the peak \( S_0 \) amplitude while using shear potential only.

vi. If the AE source is located at top surface, the effect of the excitation pressure and shear potentials on the \( A_0 \) and \( S_0 \) modes seems to be decoupled: pressure potential does not contribute to \( S_0 \) and bulk wave amplitude whereas shear potential does not contribute to the \( A_0 \) wave amplitude.

vii. For top-surface AE source, pressure excitation potential has contribution to the high amplitude low-frequency component of the \( A_0 \) wave packet. This contribution decreases as the source depth increases.

viii. Higher peak time has the contribution to the low frequency components of resulting AE signal.

5.3. Future work

Substantial future work is still needed to verify the hypotheses and substantiate the calculation of the AE source potentials that produce the elastic wave excitation. The extensive experimental AE monitoring data existing in the literature should be explored to find actual physical signals that could be compared with numerical predictions in order to extract factual data about the amplitude and time-evolution of the AE source potentials. A frequency analysis of time domain signal should be done to analyze the frequency content of the captured AE signals. Frequency content may help to distinguish different source types and source location. If necessary, additional experiments with wider band AE sensors should be conducted. An inverse algorithm could then be developed to characterize the AE source during crack propagation. The source characterization can provide information about the amount of energy released from the crack by analyzing the waveform by deconvolution process. Therefore, it may help to generate a qualitative as well quantitative description of the crack propagation phenomenon. In addition, multiple sensors can be used to localize the AE events, which may help for NDE or SHM applications. A further extensive study on the effect of plate thickness, AE source peak time, and AE source depth on AE elastic waves would be recommended. A further extensive study on the synergistic effect of rise time and source depth would be recommended. The contribution of pressure and shear excitation potentials to the total energy release from a crack needs to be studied through types of materials and crack types. A non-axisymmetric of AE elastic wave generation due to excitation potentials need to be investigated.

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Appendix A. Wave equations for potentials

Using the Eq. (3) and (6), into Eq. (2),

\[
(\lambda + \mu) \text{grad div} (\text{grad} \Phi) + (\lambda + \mu) \text{grad div} (\text{curl} \vec{H}) - \mu \text{curl curl} (\text{grad} \Phi) - \mu \text{curl curl} (\text{curl} \vec{H}) + \rho (\text{grad} A^* + \text{curl} B^*) = \rho (\text{grad} \Phi + \text{curl} \vec{H}) \tag{A1}
\]

Here,

\[
\text{grad div} (\text{curl} \vec{H}) = 0 \text{ and curl(curl(grad)Φ)=0}
\]

Therefore, Eq. (A1) becomes,

\[
(\lambda + \mu) \text{grad div} (\text{grad} \Phi) - \mu \text{curl curl} (\text{curl} \vec{H}) + \rho (\text{grad} A^* + \text{curl} B^*) = \rho (\text{grad} \Phi + \text{curl} \vec{H}) \tag{A2}
\]

Upon rearranging of Eq. (A2)

\[
\text{grad}[(\lambda + \mu) \text{ div grad} \Phi + \rho A^* - \rho \Phi)] - \text{curl}[\mu \text{ curl curl} \vec{H} - \rho B^* + \rho \vec{H}] = 0 \tag{A3}
\]

Eq. (A3) to hold at any place and any time, the components in parentheses must be independently zero, i.e.,

\[
(\lambda + \mu) \text{ div grad} \Phi + \rho A^* - \rho \Phi = 0 \tag{A4}
\]

\[
\mu \text{ curl curl} \vec{H} - \rho B^* + \rho \vec{H} = 0 \tag{A5}
\]

Upon division by \( \rho \) and rearrangement,

\[
c^2_p \text{ div grad} \Phi + A^* - \Phi = 0 \tag{A6}
\]

\[
c^2_p \text{curl curl} \vec{H} - B^* + \vec{H} = 0 \tag{A7}
\]

Here, \( c^2_p = \frac{\lambda + 2\mu}{\rho} \), \( c^2_s = \frac{\mu}{\rho} \)

Appendix B. Fourier transform in z direction

The evaluation of the integral in Eqs. (72) and (73) can be done by the residue theorem using a close contour. Since the poles are on the real axis the integral equal to

\[
\int_{-\infty}^{\infty} \frac{1}{\xi^2 - \xi^2_p} e^{i(\xi-z) \zeta} d\xi = \pi i \sum \text{Res}(f(\zeta); \zeta_j) \tag{B1}
\]

\[
\text{Res}(f(\zeta_j)) = (\zeta - \zeta_j) f(\zeta_j) \tag{B2}
\]

Here \( \zeta \) are the poles of \( \zeta \) on real axis. Poles of \( \zeta \) for Eq. (70) are

\[
\zeta = \pm \xi \rho \tag{B3}
\]

To evaluate the integration of Eq. (72) two cases are considered as follow,

Case 1: \( z - z_0 > 0 \)

\[
\tilde{\Phi}_1(\xi, y) = \frac{A}{4\pi^2} \int_{-\infty}^{\infty} \frac{1}{\xi^2 - \xi^2_p} e^{i(\xi-z) \zeta} d\xi \tag{B4}
\]
To evaluate the contour integration an arc at the infinity on the upper half plane (Fig. B1) is added because the function converges in upper half circle.

According to residue theorem the integrant is,

$$\int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \zeta_p} e^{i\zeta (z - z_0)} d\zeta = \pi i \sum_{\text{real axis}} \text{Res}(f(\zeta): \zeta_j)$$

(B5)

Here $\zeta_j$ are the poles of $f(\zeta) = \frac{1}{\zeta^2} e^{-i\zeta z_0} e^{i\zeta z}$

Using the poles, residue theorem gives,

$$\int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \zeta_p} e^{i\zeta (z - z_0)} d\zeta$$

$$= \pi i \left[ \left( \zeta - \zeta_p \right) \left( \zeta + \zeta_p \right) e^{i\zeta (z - z_0)} \right]_{\zeta = \zeta_p}$$

$$+ \pi i \left[ \left( \zeta + \zeta_p \right) \left( \zeta + \zeta_p \right) e^{i\zeta (z - z_0)} \right]_{\zeta = -\zeta_p}$$

(B6)

Upon rearrangement,

$$\int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \zeta_p} e^{i\zeta (z - z_0)} d\zeta = -\frac{\pi}{\eta_p} \left( \frac{e^{i\eta_p (z - z_0)}}{2i} - e^{-i\eta_p (z - z_0)} \right)$$

(B7)

Using Euler formula in Eq. (B7), i.e.,

$$\int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \zeta_p} e^{i\zeta (z - z_0)} d\zeta = -\frac{\pi}{\eta_p} \sin \eta_p (z - z_0)$$

(B8)

Case 2: $z - z_0 < 0$

For $z - z_0 < 0$, $z - z_0 = -|z - z_0|

\phi_1(\xi, y) = \frac{A}{4\pi^2} \int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \zeta_p} e^{-i\xi (|z - z_0|)} d\zeta$$

(B9)

This time an arc at the infinity on the upper half plane cannot be added because the function does not converge in upper half circle. To evaluate the contour integration of above function an arc at the infinity on the lower half plane (Fig. B2) can be added because the function converges in lower half circle.

$$\int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \zeta_p} e^{-i\xi (|z - z_0|)} d\zeta = -\pi i \sum_{\text{real axis}} \text{Res}(f(\zeta): \zeta_j)$$

(B10)

Using the poles, residue theorem gives,

$$\int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \zeta_p} e^{-i\xi (|z - z_0|)} d\zeta$$

$$= -\pi i \left[ \left( \zeta - \zeta_p \right) \left( \zeta + \zeta_p \right) e^{-i\xi (|z - z_0|)} \right]_{\zeta = \zeta_p}$$

$$- \pi i \left[ \left( \zeta + \zeta_p \right) \left( \zeta + \zeta_p \right) e^{-i\xi (|z - z_0|)} \right]_{\zeta = -\zeta_p}$$

(B11)

Upon rearrangement,

$$\int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \zeta_p} e^{-i\xi (|z - z_0|)} d\zeta = -\frac{\pi}{\zeta_p} \left( \frac{e^{i\zeta_p (|z - z_0|)}}{2i} - e^{-i\zeta_p (|z - z_0|)} \right)$$

(B12)

Using Euler formula in Eq. (B12), i.e.,

$$\int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \zeta_p} e^{-i\xi (|z - z_0|)} d\zeta = -\frac{\pi}{\zeta_p} \sin \zeta_p |z - z_0|$$

(B13)

By observing both cases the following summary can be made: For $y > 0$

$$\int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \zeta_p} e^{i\zeta (z - z_0)} d\zeta = -\frac{\pi}{\zeta_p} \sin \zeta_p |z - z_0|$$

(B14)

For $y < 0$

$$\int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \zeta_p} e^{-i\xi (|z - z_0|)} d\zeta = -\frac{\pi}{\zeta_p} \sin \zeta_p |z - z_0|$$

(B15)

Therefore, whether, $z - z_0$ is positive or negative, the contour integral result is same. Integration formula for Eq. (72) can be written as

$$\hat{\Phi}_1(\xi, z) = \frac{A}{4\pi^2} \int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \zeta_p} e^{i\zeta (z - z_0)}$$

(B16)

Similarly from Eq. (71), the evaluation of the integral is as follow,

$$\tilde{H}_{B1}(\xi, z) = \frac{\xi B_0}{8\pi^2} \int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \zeta_p} e^{i\zeta (z - z_0)}$$

(B17)

Appendix C. Hankel transform in x direction

The integrant in Eq. (132) is singular at the roots $D_s$ and $D_h$, i.e.,

$$D_s = 0$$

(C1)

$$D_h = 0$$

(C2)

The Eqs. (C1) and (C2) are the Rayleigh-Lamb equations for symmetric and anti-symmetric modes. The evaluation of integral can be done by residue theorem. In order to use residue theorem by close contour limit of the integral should be $-\infty$ to $\infty$. This aspect is obtained through the contour unfolding method described by ref 0 pp 613–614.

Let,

$$f(\xi) = \left( \frac{N_c}{D_s} + \frac{N_h}{D_h} \right)$$

(C3)
The function $f(\xi)$ of Eq. (3) is an odd function i.e., $f(-\xi) = -f(\xi)$; hence Eq. (132) become
\[
I_1 = \frac{1}{2} \int_{-\infty}^{\infty} \left( f(\xi) H_0^{(1)}(\xi r) \right) d\xi
\]  
(C4)

Here, $H_0^{(1)}$ is the Hankel transform of the first kind and order 0.

The evaluation of integral in Eq. (C4) can be done by residue theorem by using a close contour as shown in Fig. C1. The positive roots correspond to forward propagating waves. To satisfy the radiation boundary condition at $x = \infty$, that is no incoming waves from infinity, the negative real poles are avoided in the contour($x > 0$). The integration of Eq. (129) can be written as the sum of the residues,
\[
I_1 = \frac{1}{2i} \sum_{k=1}^{\infty} \text{Res}(\xi_k)
\]  
(C5)

For the poles $\xi_k$, Eq. (C5) becomes
\[
I_1 = \pi i \sum_{k=1}^{\infty} \text{Res}(\xi_k)
\]  
(C6)

From residue theorem, if $f(z) = \frac{N(z)}{D(z)}$ and $z = a$ is a simple pole then$\text{Res}(a) = \lim_{z \to a} \frac{N(z)}{D(z)}$, where $D'(a) = \frac{D(a)}{d} |_{z=a}$. Applying residue theorem to obtain displacement equation
\[
y_u = \pi i \left( \sum_{j=1}^{k} \left[ P_k(\xi_j) N_k(\xi_j^2) \right] H_0^{(1)}(\xi_j^2 r) \right)
\]  
\[+ \sum_{j=0}^{k} \left[ P_k(\xi_j^2) N_k(\xi_j^2) \right] H_0^{(1)}(\xi_j^2 r) \]  
(C7)

Here
\[
D'(\xi_j^2) = \frac{4\xi_j^2}{\xi_j^2 - c_p^2 \xi_j^2} \left[ \xi_j^2 (\xi_j^2 - c_p^2) - 2(\xi_j^2 c_p^2) \right] \cos(\xi_j d) \sin(\xi_j d) - d \xi_j^2 (\xi_j^2 - c_p^2) \sin(\xi_j d) \sin(\xi_j d)
\]  
\[+ d \xi_j^2 (\xi_j^2 - c_p^2) \cos(\xi_j d) \cos(\xi_j d) - 8\xi_j^2 (\xi_j^2 - c_p^2) \sin(\xi_j d) \cos(\xi_j d) \]  
(C8)

The summation of Eq. (C7) is taken over the symmetric and anti-symmetric positive real wavenumbers $\xi_j$ and $\xi_j^*$. At given frequency, there are $j = 0, 1, 2, ..., j$ symmetric Lamb wave modes and $j = 0, 1, 2, ..., j$ anti-symmetric Lamb wave modes.

Upon substitution of Eq. (58) into Eq. (133); we get
\[
I_2 = \int_{0}^{\infty} \frac{A}{4\pi^2} \cos \left( \frac{1}{\xi} \right) \left( \frac{d_1}{C_1} \sqrt{\left( \frac{\omega}{C_p} \right)^2 - \xi^2} \right) J_0(\xi r) d\xi
\]  
(C10)

Apply integration by parts to Eq. (C10)
\[
u = \xi_0 J_0(\xi r);
\]  
\[
f \int uvd\xi = u \int vd\xi - \int \left( \frac{du}{dx} \right) \int vd\xi d\xi
\]  
(C11)

After integrating by parts using Eq. (C11), Eq. (C11) becomes
\[
I_2 = \frac{A}{4\pi} \int_{0}^{\infty} \frac{\xi^2}{\sqrt{\left( \frac{\omega}{C_p} \right)^2 - \xi^2}} J_1(\xi r) d\xi
\]  
(C12)

Only the positive wavenumbers correspond to forward propagating waves will be considered here. For positive real valued wave number the integration of Eq. (C12) can be determined using inverse transform of ref Erdélyi, 1954), pp 35 ((23) for $v = 1$, i.e.,
\[
I_2 = \frac{A}{4\sqrt{2\pi}} \int_{0}^{\infty} \frac{\xi^3}{\sqrt{\left( \frac{\omega}{C_p} \right)^2 - \xi^2}} J_1(\xi r) d\xi
\]  
(C13)

Here $Y_{1}(\xi r)$ is the Bessel function of second kind.
Recall Eq. (134)
\[
I_3 = \frac{B_0}{8\pi} \int_{0}^{\infty} \frac{\xi^3}{2C_1} \sin(\xi d_1) J_0(\xi r) d\xi
\]  
(C14)

Using Eq. (58) into Eq. (C14) yields
\[
I_3 = \frac{B_0}{8\pi} \int_{0}^{\infty} \frac{\xi^3}{2\sqrt{\left( \frac{\omega}{C_p} \right)^2 - \xi^2}} J_0(\xi r) d\xi
\]  
(C15)

Upon rearrangement of Eq. (C15) in the following way
\[
I_3 = \frac{B_0}{8\pi} \int_{0}^{\infty} \frac{\xi^3}{2\sqrt{\left( \frac{\omega}{C_p} \right)^2 - \xi^2}} J_0(\xi r) d\xi
\]  
(C16)
Using general formula presented by reference Erdélyi, 1954), pp 5 (3) into Eq. (C16), for \( m = 1 \) and \( \nu = 0 \):

\[
I_1 = \frac{B_0}{8\pi} \frac{1}{d^3} \int_0^1 \frac{\xi^{3/2} \sin \left( \frac{d_1 \sqrt{(\omega/c_1)^2 - \xi^2}}{\sqrt{(\omega/c_1)^2 - \xi^2}^2} \right)}{\sqrt{(\omega/c_1)^2 - \xi^2}^2} J_1(\xi \tau) \left( \xi \tau \right)^{1/2} d\xi \tag{C17}
\]

Considering the integral part of Eq. (C17), i.e.,

\[
\int_0^1 \frac{\xi^{3/2} \sin \left( \frac{d_1 \sqrt{(\omega/c_1)^2 - \xi^2}}{\sqrt{(\omega/c_1)^2 - \xi^2}^2} \right)}{\sqrt{(\omega/c_1)^2 - \xi^2}^2} J_1(\xi \tau) d\xi
\]

For positive real wave number value the integration of Eq. (C18) can be determined by using formula presented by ref. Erdélyi, 1954) pp 35 (23) for \( \nu = 1 \), i.e.,

\[
\int_0^1 \frac{\xi^{3/2} \sin \left( \frac{d_1 \sqrt{(\omega/c_1)^2 - \xi^2}}{\sqrt{(\omega/c_1)^2 - \xi^2}^2} \right)}{\sqrt{(\omega/c_1)^2 - \xi^2}^2} J_1(\xi \tau) d\xi
\]

Here \( \nu \) is the Bessel function of second kind. Using Eq. (C19) into Eq. (C17), we get

\[
I_1 = \frac{B_0}{8\sqrt{2\pi}} \left( \frac{\omega/c_1}{d_1^3} \right) \left( d_2^2 + r^2 \right)^{3/4} \left( r^2 \right)^{1/2}
\]

\[
\left[ Y_1/2 \left( \frac{\omega/c_1 (d_2^2 + r^2)^{1/2}}{2\omega/c_1 (d_2^2 + r^2)^{1/2}} \right)^2 \right]
\]

\[
\times \left[ -\frac{3}{2\omega/c_1 (d_2^2 + r^2)^{1/2}} Y_{5/2} \left( \frac{\omega/c_1 (d_2^2 + r^2)^{1/2}}{2\omega/c_1 (d_2^2 + r^2)^{1/2}} \right)^2 \right]
\]

\[
\times \left[ -\frac{3}{2\omega/c_1 (d_2^2 + r^2)^{1/2}} Y_{3/2} \left( \frac{\omega/c_1 (d_2^2 + r^2)^{1/2}}{2\omega/c_1 (d_2^2 + r^2)^{1/2}} \right)^2 \right]
\]

\[
\times \left[ -\frac{3}{2\omega/c_1 (d_2^2 + r^2)^{1/2}} Y_{1/2} \left( \frac{\omega/c_1 (d_2^2 + r^2)^{1/2}}{2\omega/c_1 (d_2^2 + r^2)^{1/2}} \right)^2 \right]
\]

References


