A comparative convergence and accuracy study of composite guided-ultrasonic wave solution methods: Comparing the unified analytic method, SAFE method and DISPERSE

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Abstract
Multiple approaches and programs are available to the public to predict ultrasonic guided-wave propagation dispersion curves in a material. Each approach and program will have its own advantage and disadvantage making it suitable for a specific end use and less suitable for others. This manuscript aims to compare three different guided-ultrasonic wave dispersion curves retrieval methods for multiple cases. First a single layer of isotropic, unidirectional, orthotropic and monoclinic material is examined, followed by a multi-layer case (consisting of 10 layers of the aforementioned materials) and finally four different laminates and one sandwich laminate are evaluated. The goal of this manuscript is to give a concise overview of the advantages and limitations of each approach to assist the end user in choosing the right program to use.

Keywords
Guided-ultrasonic wave dispersion curves, structural health monitoring, isotropic, anisotropic, composite laminates, unified analytic method, SAFE, DISPERSE

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Introduction
Traditional non-destructive evaluation (NDE) methods were developed with isotropic materials in mind, therefore for newer aircraft (like the Boeing B787 and the Airbus A350XWB), where the fuselage is made out of composite materials, these NDE methods do not yield accurate results. In the previous decades, focus has shifted towards advances in non-destructive tests, and evaluation and inspection methods for aerospace composite structures.1,2 Within the NDE research field methods, utilizing ultrasonic guided-wave (GW) propagation, especially Lamb waves,3 are preferred due to small amplitude attenuation over long distances.4,5 Examples in literature utilizing Lamb waves to detect, localize, or quantify damage in composites are abundant.6–11

Accurate knowledge regarding the dispersion curves in anisotropic material can lead to an increase in the probability of damage detection.12 Even though wave propagation in isotropic and anisotropic materials is discussed in detail in the literature, there remains a lack of accurate prediction programs that are applicable to both isotropic and anisotropic material at the same time.13–18 Part of the problem originates from the decoupling of Lamb waves and shear horizontal waves and the use of different algorithms depending on the material type.

For composite structures which consist of N-layers, the interface conditions (displacement continuity through the thickness), stress balance for adjacent layers and the traction free boundary condition on the outer surfaces of the structure are required to be satisfied. The first method developed to address this problem was the transfer matrix method (TMM) in the 1950s.19,20 The TMM was initially developed for
isotropic materials. However, it was extended by Nayfeh to incorporate anisotropic \( N \)-layered composites as well.\textsuperscript{16} In the TMM, displacements and stresses in the \( i \)th layer are only described in terms of displacement and stresses of adjacent neighboring layers: the layer above \( i+1 \) and the layer below \( i-1 \). A clear advantage of the TMM is the compact transfer matrix that is obtained in which all the layers are described in terms of the boundary condition at the surfaces of the material. The TMM, therefore, results in a \( 6 \times 6 \) matrix regardless of the number of layers in the laminate. Numeric instabilities, however, can occur when dealing with high frequencies, large thicknesses, or a combination of the two.

A different method called the global matrix method (GMM) was developed to overcome the numeric instabilities that occurred for the TMM.\textsuperscript{21} This method assembles the displacement and stress boundary conditions for all layers into one global matrix. This results, however, in an increasing global matrix when increasing the number of layers in a laminate: a \( 6N \times 6N \) matrix where \( N \) is the number of layers. However, Mal provided an alternative procedure to deal with the TMM instabilities that lead to a banded \( Q \)-matrix with dimensions of \( 5N \times 6N \).\textsuperscript{22} The \( Q \)-matrix element are finite and therefore the system of \( 6N \) equations was solved without any difficulties.

Rokhlin and Wang also investigated the numeric instabilities of the TMM and introduced the stiffness matrix method (SMM) as a result.\textsuperscript{23,24} Similar to the TMM, the SMM applies a recursive algorithm to obtain a compact matrix. The SMM, however, relates the stresses at the surfaces to the displacements at the surfaces through the layer stiffness matrix and it is considered to be unconditionally stable.

Based on the GMM, Pavlakovic et al. introduced a dispersion curve algorithm called DISPERSE which is applicable to both isotropic and composite materials.\textsuperscript{25,26} The GMM has also been applied to investigate the wave propagation time histories in both isotropic and cross-ply laminates.\textsuperscript{27–29}

Bartoli et al. proposed a semi-analytical finite element (SAFE) approach to obtain the dispersion curves in an arbitrary cross-section that accounted for viscoelastic material damping.\textsuperscript{30} Dispersion curves can be obtained in both isotropic and anisotropic material using the SAFE approach. However, it requires the implementation of a 1D finite element method (FEM). Wang and Yuan reported on the formulation of Lamb waves in composites using 3D elasticity theory.\textsuperscript{31} The reported formulation is mathematically complicated, but able to predict the dispersion in \( N \)-layered composites. All the experimentally tested laminates were, however, quasi-isotropic laminates.

Another method, investigated by Gopalakrishnan,\textsuperscript{32} used the two dimension spectral FEM to retrieve the Lamb wave dispersion curves for laminated composites structures. However, the shear-horizontal wave modes were not retrieved.

The structure of this manuscript is as follows:

(a) the theoretical background of the three investigated methods are briefly discussed;
(b) the dispersion curves for multiple cases (single layer, multi layer, laminates and a sandwich structure) are presented and compared;
(c) as last a summary and discussion is given to conclude the manuscript.

**Theoretical recap**

In this section a recap of the theory for the unified analytic method (UAM), the SAFE method and DISPERSE is provided for convenience. For more details on the UAM, the SAFE method and DISPERSE the reader is encouraged to read cited texts.\textsuperscript{17,25,30,33–35}

**The unified analytic method (UAM)**

In the UAM, an ultrasonic GW propagating in an arbitrary media is described by its particle displacement, which is a function of the angular frequency \( \omega \), wavenumber \( \xi \) and phase velocity \( v = \omega / \xi \). The wave consists of a superposition of waves propagating in multiple directions, i.e. a wave propagating in the \( x_1 \) direction will be the result of a superposition of waves propagating in the \( x_1 \) and \( x_3 \) direction under \( x_2 \) invariant condition, as can be seen by

\[
\mathbf{u} = \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{bmatrix} e^{(k_1 x_1 + k_3 x_3 + i \alpha t)} = \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{bmatrix} e^{i(\xi x_1 + \alpha x_3 - vt)}
\]

(1)

where

\[
k_1 = \xi \quad k_3 = \alpha \xi
\]

(2)

Note, \( k_1 \) and \( k_3 \) in equation (1) are the directional wavenumbers, and \( \alpha \) is ratio between the wavenumbers in the \( x_3 \) and \( x_1 \) direction, \( \alpha = k_3 / k_1 \). In addition, recall the equation of motion as

\[
c_{ijkl} \mathbf{u}_{m,l} = \rho \ddot{u}_i \quad i, j, m, l = 1, 2, 3
\]

(3)

Substituting the derivatives of the displacements, equation (1), with respect to the spatial variables.
and time variable into equation (3), rearranging and dividing out common factors yields

\[
\begin{bmatrix}
(C_{11} - \rho v^2) + C_{55} \alpha^2 & C_{16} + C_{45} \alpha^2 & (C_{13} + C_{55}) \alpha \\
C_{16} + C_{45} \alpha^2 & (C_{66} - \rho v^2) + C_{44} \alpha^2 & (C_{36} + C_{45}) \alpha \\
(C_{13} + C_{55}) \alpha & (C_{36} + C_{45}) \alpha & (C_{55} - \rho v^2) + C_{33} \alpha^2
\end{bmatrix}
\begin{bmatrix}
\hat{u}_1 \\
\hat{u}_2 \\
\hat{u}_3
\end{bmatrix}
= 0
\]  

(4)

Applying the zero-determinant condition to equation (4), and expanding its determinant resulted in the bi-cubic equation, which after solving yielded the eigenvalues, \( \alpha \)

\[
A_6 \alpha^6 + A_4 \alpha^4 + A_2 \alpha^2 + A_0 = 0
\]

\( (\alpha^2)^3 + A'_4(\alpha^2)^2 + A'_2(\alpha^2) + A'_0 = 0 \)  

(5)

It is important to note that in the second form of equation (5) the coefficients are normalized with respect to \( A_6 \). The bi-cubic coefficients \( A_6, A_4, A_2 \) and \( A_0 \) are represented below as a function of the material’s stiffness matrix, density and phase velocity

\[
A_6 = C_{33} C_{44} C_{55} - C_{33} C_{45}^2
\]

\[
A_4 = (C_{44} C_{55} - C_{45}^2)(C_{55} - \rho v^2) + C_{33} C_{55}(C_{66} - \rho v^2)
\]

\[
+ C_{55} C_{44}(C_{11} - \rho v^2) - C_{16} C_{45} C_{33}
\]

\[
+ 2(C_{36} + C_{43})(C_{13} + C_{55}) C_{45}
\]

\[
-(C_{13} + C_{55}^2) C_{44} - (C_{45} + C_{36})^2 C_{55}
\]

\[
A_2 = C_{33}(C_{11} - \rho v^2)(C_{66} - \rho v^2) + C_{44}(C_{11} - \rho v^2)
\]

\[
\times (C_{55} - \rho v^2) + C_{55}(C_{66} - \rho v^2)(C_{55} - \rho v^2)
\]

\[-(C_{11} - \rho v^2)(C_{45} + C_{36})^2 - (C_{66} - \rho v^2)
\]

\[
\times (C_{13} + C_{55}^2) - 2(C_{55} - \rho v^2)(C_{55} - \rho v^2)
\]

\[+ 2 C_{16}(C_{45} + C_{36})(C_{13} + C_{55}) - C_{16}^2 C_{33}
\]

\[
A_0 = [(C_{11} - \rho v^2)(C_{66} - \rho v^2) - C_{16}^2](C_{55} - \rho v^2)
\]

(6)

Subsequently the eigenvalues were used to obtain the eigenvectors, \( \mathbf{U} \), after which the displacement field matrix \( \mathbf{B}' \) was formulated as

\[
\mathbf{B}'(x_3) = \left[ \mathbf{b}^{(1)}(x_3) \mathbf{b}^{(2)}(x_3) \mathbf{b}^{(3)}(x_3) \mathbf{b}^{(4)}(x_3) \mathbf{b}^{(5)}(x_3) \mathbf{b}^{(6)}(x_3) \right]
\]

(7)

where

\[
\mathbf{b}^{(j)}(x_3) = \mathbf{U}^{(j)} e^{i \mathbf{e}^{(j)}(x_3)} \quad j = 1, 2, \ldots, 6
\]

To reconstruct the complete wave, the eigenvalues and corresponding eigenvectors were substituting

\[
\mathbf{u} = \hat{\mathbf{u}}(x_3)e^{i(x_3 - vt)} = \left( \sum_{j=1}^{n} \eta_j \mathbf{U}^{(j)} e^{i \mathbf{e}^{(j)}(x_3)} \right) e^{i(x_3 - vt)}
\]

(9)

where \( \eta \) are the partial-wave participation factors. The \( x_3 \)-dependent, thickness, part was isolated to give

\[
\hat{\mathbf{u}}(x_3) = \mathbf{B}'(x_3) \eta
\]

(10)

In a similar fashion the stress field matrix \( \mathbf{B}' \) was derived; the stress-displacement relationship under \( x_2 \)-invariant conditions for a monoclinic lamina was used as given by

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{31} \\
\sigma_{12}
\end{bmatrix}
= \mathbf{C}
\begin{bmatrix}
u_{1,1} \\
u_{3,3} \\
u_{2,3} \\
u_{3,1} + u_{1,3} \\
u_{1,1}
\end{bmatrix}
\]

(11)

where \( \mathbf{C} \) is

\[
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\
C_{22} & C_{23} & 0 & 0 & C_{26} & C_{33} & 0 & 0 & C_{36} \\
C_{44} & C_{45} & 0 & C_{55} & 0 & \sym
\end{bmatrix}
\]

(12)

For GW propagation in an arbitrary media the upper and lower surfaces are traction free. The normal vector on the upper and lower surface is \( \hat{\mathbf{e}}_3 \), therefore, the traction on the surface will consist of the normal stress \( \sigma_{33} \) and the two shear stresses \( \sigma_{23} \) and \( \sigma_{31} \). Evaluating equation (11) by taking the stresses which were influenced by the traction on the upper and lower surface yielded

\[
\sigma_{33} = C_{13}u_{1,1} + C_{33}u_{3,3} + C_{16}u_{2,1}
\]

\[
\sigma_{23} = C_{44}u_{2,3} + C_{45}(u_{3,1} + u_{1,3})
\]

\[
\sigma_{31} = C_{45}u_{2,3} + C_{55}(u_{3,1} + u_{1,3})
\]

(13)
Substituting the displacement derivations obtained from equation (9) into equation (13), rearranging and collecting the stress vector yielded

\[ \hat{\sigma}(x_3) = \begin{pmatrix} \hat{\sigma}_{33} \\ \hat{\sigma}_{23} \\ \hat{\sigma}_{13} \end{pmatrix} = B'(x_3)\eta \]

(14)

where \( B' \) and \( b^{(j)} \) is given by

\[ B'(x_3) = \begin{bmatrix} b^{(1)}(x_3) & b^{(2)}(x_3) & b^{(3)}(x_3) & b^{(4)}(x_3) & b^{(5)}(x_3) & b^{(6)}(x_3) \end{bmatrix} \]

\[ b^{(j)}(x_3) = i\xi \begin{bmatrix} C_{13}U_{1}^{(j)} + C_{33}\alpha^{(j)}U_{3}^{(j)} + C_{36}U_{2}^{(j)} \\ C_{43}\alpha^{(j)}U_{2}^{(j)} + C_{45}\alpha^{(j)}U_{3}^{(j)} + C_{45}\alpha^{(j)}U_{1}^{(j)} \\ C_{53}\alpha^{(j)}U_{2}^{(j)} + C_{55}\alpha^{(j)}U_{3}^{(j)} + C_{55}\alpha^{(j)}U_{1}^{(j)} \end{bmatrix} e^{i\alpha^{(j)}x_1} \]

(15)

Finally, the dispersion curves were retrieved by applying the traction free boundary conditions, stresses at the top and bottom of the medium are zero, and concatenating the two stress field matrices into one to obtain

\[ \begin{bmatrix} B'(0) \\ B'(h) \end{bmatrix} \eta = D\eta = 0 \]

(16)

The UAM formulation for the dispersion curves was defined in the wavenumber-phase velocity solution space in the determinant of the \( D \)-matrix. Both the \( D \)-matrix and its corresponding determinant are complex, therefore, a sign change occurs if and only if both the real and imaginary part of the complex number change sign simultaneously. The phase approach, as presented by Barazanchy and Giurgiutiu, was used to retrieve the sign changes in the solution space.\(^{33,34}\)

For multi-layered media the GMM, TMM or SMM can be applied to obtain the multi-layered \( D \)-matrix. In this manuscript the results obtained using the TMM are shown.

**Semi analytic finite element (SAFE) method**

The SAFE method requires discretization in the thickness direction (the \( y-z \) plane) to describe the mode-shapes of the GW propagating in the \( x \)-direction with wavenumber \( \xi \) at frequency \( \omega \)

\[ u = [u_x, u_y, u_z]^T \]

\[ \sigma = [\sigma_x, \sigma_y, \sigma_z, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}]^T \]

\[ \epsilon = [\epsilon_x, \epsilon_y, \epsilon_z, \epsilon_{xy}, \epsilon_{xz}, \epsilon_{yz}]^T \]

(17)

Using the displacement, stress and strain field components described in equation (17) the stress-displacement relation \( (\sigma = Ce) \) can be written in matrix form as

\[ \epsilon = \begin{bmatrix} L_x \frac{\partial}{\partial x} + L_y \frac{\partial}{\partial y} + L_z \frac{\partial}{\partial z} \end{bmatrix} u \]

(18)

where

\[ L_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad L_y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad L_z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \]

(19)

**Equation of motion.** Hamilton’s equation is utilized to formulate the equation of motion in the cross-section; its variation from is expressed as

\[ \delta H = \int_{t_1}^{t_2} \delta(\Phi - K) dt = 0 \]

(20)

where the strain energy \( \Phi \) and kinetic energy \( K \) are given by

\[ \Phi = \frac{1}{2} \int_V \epsilon^T C \epsilon \, dV, \quad K = \frac{1}{2} \int_V \dot{u}^T \rho \dot{u} \, dV \]

(21)

The density, displacement time derivative and volume are represented by \( \rho \), \( u \) and \( V \), respectively

\[ \int_{t_1}^{t_2} \left[ \int_V \delta(\epsilon^T)C\epsilon \, dV + \int_V \delta(u^T)\rho \dot{u} \, dV \right] dt \]

(22)

Equation (22) is obtained after applying integration by parts to equation (20). Furthermore, the displacement is assumed to be harmonic along the propagation direction, hence

\[ u(x,y,z,t) = \begin{bmatrix} u_x(x,y,z,t) \\ u_y(x,y,z,t) \\ u_z(x,y,z,t) \end{bmatrix} = \begin{bmatrix} U_x(y,z) \\ U_y(y,z) \\ U_z(y,z) \end{bmatrix} e^{i(\xi - \omega t)} \]

(23)
**Finite element method.** Equation (23) is rewritten in its discretized form as a function of the shape functions \( N_k(y,z) \) and unknown nodal displacements \( U_{xk}, U_{yk} \) and \( U_{zk} \)

\[
\mathbf{u}^{(e)}(x,y,z,t) = \sum_{k=1}^{n} N_k(y,z) U_{xk} e^{\delta(x-\omega t)}
\]

and

\[
N(y,z) = \begin{bmatrix} N_1 & N_2 & \cdots & N_n \\ N_1 & N_2 & \cdots & N_n \\ \vdots & \vdots & \ddots & \vdots \\ N_1 & N_2 & \cdots & N_n \end{bmatrix}
\]

(25)

and

\[
q^{(e)} = [U_{x1} \; U_{y1} \; U_{x2} \; U_{y2} \; U_{x3} \; \ldots \; U_{xn} \; U_{yn} \; U_{zn}]
\]

(26)

where \( n \) denotes the number of nodes per element. Subsequently, the strain vector in the element can be written as function of the unknown nodal displacements

\[
e^{(e)} = \left[ L_x \frac{\partial}{\partial x} + L_y \frac{\partial}{\partial y} + L_z \frac{\partial}{\partial z} \right] N(y,z) q^{(e)} e^{\delta(x-\omega t)}
\]

(27)

where \( L \) is given in equation (19), \( B_1 \) and \( B_2 \) are represented as

\[
B_1 = L_x \frac{\partial N}{\partial y} + L_z \frac{\partial N}{\partial z} \quad B_2 = L_x N
\]

(28)

The discrete form of Hamilton's equation in the cross-section direction is now formulated as a function of the number of elements \((n_e)\) in that direction. Therefore, equation (22) becomes

\[
\int_{t_1}^{t_2} \sum_{e=1}^{n_e} \left[ \int_Y \delta(e^{(e)} \mathbf{C} e^{(e)}) dV_e \\
+ \int_Y \delta(q^{(e)}) du^{(e)} dV_e \right] dt = 0
\]

(29)

where the subscript \( e \) denotes the element number

\[
\int_Y \delta(e^{(e)} \mathbf{C} e^{(e)}) dV_e = \delta q^{(e)^T}
\]

\[
\int_{\Omega_e} \left[ B_1^T C B_1 - i e B_1^T C B_1 + i e B_1^T C B_2 + \xi^2 B_2^T C B_2 \right] du^{(e)}
\]

(30)

Equation (30) was obtained by substituting equation (27) into equation (29)

\[
\int_Y \delta(u^{(e)^T}) u^{(e)} dV_e = -\omega^2 \delta q^{(e)^T} \int_{\Omega_e} N^T \rho_e N d\Omega_e q^{(e)}
\]

(31)

Similarly substituting equation (24) into equation (29) yields the kinetic energy contribution given in equation (31). Subsequently, substituting equations (30) and (31) into equation (29) yields

\[
\int_{t_1}^{t_2} \sum_{e=1}^{n_e} \delta q^{(e)^T} \left[ k_1^{(e)} + i e k_2^{(e)} + \xi^2 k_3^{(e)} - \omega^2 m^{(e)} \right] q^{(e)} dt = 0
\]

(32)

where

\[
k_1^{(e)} = \int_{\Omega_e} \left[ B_1^T C B_1 \right] d\Omega_e
\]

\[
k_2^{(e)} = \int_{\Omega_e} \left[ B_2^T C B_2 - B_1^T C B_1 \right] d\Omega_e
\]

\[
k_3^{(e)} = \int_{\Omega_e} \left[ B_2^T C B_2 \right] d\Omega_e
\]

\[
m^{(e)} = \int_{\Omega_e} N^T \rho_e N d\Omega_e
\]

(33)

Applying the FEM assembling procedure to equation (32) yields

\[
\int_{t_1}^{t_2} \delta U^T \left[ K_1 + i e K_2 + \xi^2 K_3 - \omega^2 M \right] U \right) dt = 0
\]

(34)

where \( U \) contains the unknown nodal displacements and

\[
K_1 = \bigcup_{e=1}^{n_e} k_1^{(e)} \quad K_2 = \bigcup_{e=1}^{n_e} k_2^{(e)} \quad K_3 = \bigcup_{e=1}^{n_e} k_3^{(e)} \quad M = \bigcup_{e=1}^{n_e} m^{(e)}
\]

(35)

\[
[K_1 + i e K_2 + \xi^2 K_3 + M] U = 0
\]

(36)

To obtain the non-trivial solution, Equation (34) must hold regardless the value for \( \delta U \), therefore Equation (34) reduces to the homogeneous general wave equation given in Equation (36). Equation (36) can be rewritten as a first-order eigensystem by doubling its algebraic size. In addition to the work by Bartoli et al., the SAFE method can be written as function of the wavenumber and phase velocity \( c \) instead of the frequency and wavenumber by utilizing the relationship \( (\omega = \xi c) \) between frequency, wavenumber and phase velocity

\[
\left[ \begin{array}{cc} 0 & K_1 - \omega^2 M \\
K_1 - \omega^2 M & i e K_2 \end{array} \right] \left[ \begin{array}{c} U \\
-\xi \end{array} \right] = 0
\]
The aforementioned theory was implemented using Matlab such that a direct comparison between the UAM and SAFE method was possible. The accuracy of the SAFE solution is dependent on the number of elements through the thickness, therefore a convergence study (presented in the ‘Dispersion curves’ section) was performed prior to the comparative study to ensure the accuracy of the results.

**DISPERSE**

Similar to the UAM discussed above, the Christoffel’s equation for a lamina is used as starting point in DISPERSE to retrieve the dispersion curves. The notation format used differs, however, as given by

\[
-K_1 + i\xi K_2 + \xi^2 K_3 0
\]

\[
0 0
-K_2 + i\xi K_3 + \xi^2 M
\]  

\[
U^T = 0 \quad (37)
\]

For a wave propagation in an unidirectional material along the fiber direction (DISPERSE assume wave propagation in the z-direction) equation (40) is simplified to equation (41). Where \( \xi \) is the wavenumber in the propagation direction (\( k_z \)). The matrix in equation (41) can be decoupled into two parts, a shear horizontal wave part and a Lamb wave part, the combined quasi-longitudinal and quasi-shear waves. For the Lamb waves part

\[
\zeta^2 = \frac{-B + \sqrt{B^2 - 4AC}}{2A}
\]

\[
\zeta^2 = \frac{-B - \sqrt{B^2 - 4AC}}{2A}
\]

(42)

where

\[
A = C_{66}C_{22}
\]

\[
B = -(C_{66} + C_{22})\rho\omega^2 - (C_{12}^2 + 2C_{12}C_{66} - C_{11}C_{22})\xi^2
\]

\[
C = (C_{11}\xi^2 - \rho\omega^2)(C_{66}\xi^2 - \rho\omega^2)
\]

(43)

For the shear-horizontal wave

\[
\zeta^2 = k^2(1/C_{44})(\rho\omega^2 - C_{55}\xi^2)
\]

(44)

The eigenvectors can be obtained by substituting the solutions back into Christoffel’s equation, equation (38), hence

\[
U_x = -\left(\frac{C_{11}\xi^2 + C_{66}\xi^2 - \rho\omega^2}{C_{12} + C_{66}}\right)U_z
\]

(45)

The displacements for the quasi-longitudinal waves can now be written as

\[
u_1 = \pm e(1, x)A_{1 \pm}Fe^{ikx_l}
\]

\[
u_3 = e(1, z)A_{1 \pm}Fe^{ikx_z}
\]

(46)

Similarly for the quasi-shear waves

\[
u_1 = e(2, x)A_{2 \pm}Fe^{ikx_l}
\]

\[
u_3 = \pm e(1, z)A_{2 \pm}Fe^{ikx_z}
\]

(47)

where \( F = e^{i(k_x x - \omega t)} \), \( e(a, b) \) represents the eigenvector component of \( a \) in direction \( b \) and the ± sign indicates

\[
\begin{bmatrix}
C_{11}\xi^2 + C_{66}k_x^2 + 2C_{16}\xi k_x - \rho\omega \\
C_{16}^2 + C_{26}k_x^2 + (C_{12} + C_{66})\xi k_x \\
C_{66}\xi^2 + C_{22}k_x^2 + 2C_{26}\xi k_x - \rho\omega
\end{bmatrix}
\]

\[
\begin{bmatrix}
C_{11}\xi^2 + C_{66}k_x^2 - \rho\omega \\
C_{16}^2 + C_{26}k_x^2 - \rho\omega \\
C_{66}\xi^2 + C_{22}k_x^2 - \rho\omega
\end{bmatrix}
\]

(38)

(39)

\[
\begin{bmatrix}
[l_x 0 0] \\
0 [l_y 0 0] \\
0 0 [l_z 0]
\end{bmatrix}
\]

\[
\begin{bmatrix}
l_x 0 0 \\
0 l_y 0 \\
0 l_z
\end{bmatrix}
\]

\[
\begin{bmatrix}
l_x 0 0 \\
0 l_y 0 \\
0 l_z
\end{bmatrix}
\]

\[
\begin{bmatrix}
l_x 0 0 \\
0 l_y 0 \\
0 l_z
\end{bmatrix}
\]

\[
\begin{bmatrix}
l_x 0 0 \\
0 l_y 0 \\
0 l_z
\end{bmatrix}
\]

where \( l_x, l_y \) and \( l_z \) are the directional cosines; \( v_x, v_y \) and \( v_z \) are the particle velocities; \( \rho \) is the density; \( \omega \) is the frequency; \( k \) is the wavenumber and [\( C_{ij} \)] are the stiffness constants. For anisotropic material one obtains
downwards and upwards traveling waves. The stresses are obtained as is shown equation (11). For a unidirectional material the normal and tangential stresses reduce to

\[
\begin{align*}
\sigma_{xx} &= C_{11}u_{x,x} + C_{13}u_{x,z} \\
\sigma_{xz} &= C_{52}(u_{x,z} + u_{z,x}) 
\end{align*}
\] (48)

Which after substituting equation (48) into equation (38) yields the stress for the quasi-longitudinal waves

\[
\begin{align*}
\sigma_{11} &= \pm\left[ C_{11}\xi e(1, z) + C_{12}\xi_1 e(1, x) \right] A_{1z} F e^{ikz} \\
\sigma_{13} &= \pm i C_{66}[\xi e(1, z) + \xi e(1, x)] A_{14} F e^{ikz}
\end{align*}
\] (49)

for the quasi-shear waves

\[
\begin{align*}
\sigma_{11} &= \pm\left[ C_{12}\xi e(2, z) + C_{13}\xi_1 e(2, x) \right] A_{23} F e^{ikz} \\
\sigma_{13} &= i C_{66}[\xi e(2, z) + \xi e(2, x)] A_{14} F e^{ikz}
\end{align*}
\] (50)

and for the shear-horizontal waves

\[
\sigma_{11} = \mp i C_{44}\xi_3 (\pm \xi_3) A_{3z} F e^{ikz}
\] (51)

For layered laminates the solutions provided by equations (46), (47) and (49) to (51) are combined into a layer matrix, then the GMM is applied to assemble the final global \([G]\)-matrix that has to be solved to retrieve the dispersion curves. DISPERSE is given in the phase velocity–frequency solution domain; the results therefore had to be converted to the wavenumber-phase velocity domain before a direct comparison was possible.

**Dispersion curves**

In this section the dispersion curves for four different cases are retrieved using the aforementioned methods and compared to each other. For a single layer and multi-layer material, four different material types are investigated:

(a) isotropic;
(b) unidirectional;
(c) orthotropic;
(d) monoclinic.

The stiffness matrix and corresponding density for each material are given below for completeness:

\[
\rho = 2700 \text{kg/m}^3
\] (52)

for unidirectional

\[
C = \begin{bmatrix}
143.8 & 6.2 & 6.2 & 0 & 0 & 0 \\
13.3 & 6.5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \text{ GPa},
\]

\[
\rho = 1560 \text{kg/m}^3
\] (53)

for orthotropic

\[
C = \begin{bmatrix}
70 & 23.9 & 6.2 & 0 & 0 & 0 \\
33 & 6.8 & 0 & 0 & 0 & 0 \\
14.7 & 4.2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \text{ GPa},
\]

\[
\rho = 1500 \text{kg/m}^3
\] (54)

and for the monoclinic (obtain by rotation of the unidirectional stiffness matrix by 45°)

\[
C = \begin{bmatrix}
48.1 & 36.7 & 6.35 & 0 & 0 & 32.6 \\
48.1 & 36.5 & 0 & 0 & 0 & 32.6 \\
13.3 & 0 & 0 & -0.15 & 0 & 36.1 \\
0 & 0 & 0 & 4.55 & 1.15 & 0 \\
0 & 0 & 0 & 4.55 & 1.15 & 0 \\
0 & 0 & 0 & 4.55 & 1.15 & 0
\end{bmatrix} \text{ GPa},
\]

\[
\rho = 1560 \text{kg/m}^3
\] (55)

For the laminate cases, the unidirectional material was rotated corresponding to the specific layup for each case:

(a) \([0/90]_s\);
(b) \([0/45/90]_s\);
(c) \([\pm 45/0/90]_s\);
(d) a fiber metal laminate (a combination of anisotropic and isotropic material) with a \([\pm 45/Al/90]_s\) stacking sequence.

As last, a sandwich laminate was investigated to see the effects of large thickness structures on the different ultrasonic GW algorithms.

Before comparing the results for the three different methods, it is important to note that both the UAM
and DISPERSE are based on an analytic model, SAFE however is based on a 1D FEM. The accuracy, therefore, of the SAFE solution will depend on the number of elements used in the cross-section. An initial comparison to highlight the effect of the number of elements through the thickness on the convergence of the SAFE solution (compared to the UAM) is shown in Figure 1.

It can be seen from Figure 1(a), that several higher order modes were not retrieved by the SAFE method when the number of elements through the thickness was insufficient. Figure 1(b) to (d) show that more higher order modes were retrieved when increasing the number of elements through the thickness. The accuracy of the solution, however, remained unsatisfactory. When the number of elements was increased to five satisfactory results were obtained to a certain extend. It required 15 elements through the thickness to obtain accurate results for all the higher order modes. To ensure that convergence of SAFE was met in all the cases for all the wave modes in the whole solution domain the number of elements was set to 20. This increased the computational time of SAFE significantly, however, this increase was accepted since comparing the accuracy between the different methods was considered more valuable.

Single layer media

The dispersion curves for a single layer isotropic, unidirectional, orthotropic and monoclinic material retrieved using the UAM, the SAFE method and DISPERSE are shown in Figure 2. The dispersion curves correspond to one 1 mm thick layer of each material type. It can be seen from Figure 2 that all three of the different programs performed well. However, DISPERSE had difficulties in retrieving some higher order modes as can be seen in Figure 2(c) and (d) in addition to a couple of outliers. The UAM and SAFE method were both capable of retrieving all the waves modes in the solution domain.

It is important to note, however, that DISPERSE initially had some difficulties retrieving the shear-horizontal $SH_{50}$ wave modes when using the automatic tracing option. This was solved by using the manual sweep/search option, this however relied upon some prior knowledge of the $SH_{50}$ wave mode.

Multi-layer media

The number of layers was increased from one to 10 and subsequently the layer thickness was decreased to 0.1 mm, therefore, the overall thickness remained at
1 mm which allowed for an additional comparison between the dispersion curves shown in Figures 2 and 3. The multi-layer media case was investigated to test the robustness of the UAM and to investigate the effect of multiple layers on the different solution methods. Similar to the case of a single layer both the UAM and SAFE yielded accurate results for each case. For DISPERSE, however, minor problems such as:

(a) missing wave modes;
(b) outliers were observed.

Major problems only occurred for the monoclinic test case in which the authors failed to retrieve the A0 wave mode and multiple higher order modes when using DISPERSE.

It is of great importance to note however, that this is not the usual way of modeling in DISPERSE, since a partial waves method is used instead of discretizing the waveguide. It is common practice in DISPERSE to model two or more adjacent layers of the same material as one layer with an equivalent thickness. The partial waves method allows us to represent a complex waveguide by only one layer per media; no improvement in accuracy is achieved by subdividing the media into multiple layers. For a discretized solution method such as SAFE however, dividing a material into multiple layers increases the accuracy of the solution.

Furthermore, the constant phase velocity observed in Figure 3 correspond to the bulk phase velocity of the material. The partial waves solution has a singularity at the bulk phase velocity, this singularity was accentuated when the number of layers was increased. Therefore, the bulk phase velocity are observed for the multi-layered media and not for the single layer media case.

Regardless, it is possible to tweak the convergence parameters in DISPERSE to obtain the dispersion curves without the bulk phase velocity. This, however, requires knowledgeable manual intervention in the code. In addition, DISPERSE allows for deleting incorrect solution points such as at zero wavenumber, regardless of the phase velocity.

**Laminated media**

A similar behavior was observed for laminates as for multi layered media. Both the UAM and SAFE method produced accurate results, while DISPERSE missed several higher order wave modes.

Figure 2. Dispersion curves obtained the UAM, SAFE method and DISPERSE for a single 1 mm thick layer: (a) isotropic, aluminum; (b) unidirectional CFRP; (c) orthotropic CFRP and (d) monoclinic, $\theta = 45^\circ$ CFRP.
In Figure 4(b) and (c), DISPERSE captured a dispersion curve that was not part of the solution when using the UAM and SAFE method.

In addition to conventional laminates, a fiber metal laminate with a \([\pm 45/Al/90]_s\) stacking sequence was investigated as well. This laminate consisted of isotropic/anisotropic media and the corresponding dispersion curves are shown in Figure 4(d). Again, the UAM and SAFE method provided accurate dispersion curves and DISPERSE had issues in retrieving part of the wave modes.

**Sandwich media**

Finally, a sandwich laminate (see Figure 5) was investigated to see how thicker laminates affect the accuracy of the different algorithms. The sandwich laminate consisted of two face sheets each 1.78 mm thick and a core that was 12.7 mm thick which resulted in a total thickness of 15.56 mm. The face sheets were of the same unidirectional CFRP material as discussed above, for the core the same material as discussed by Banerjee and Pol was used.\(^56\) The stiffness matrix of the core materials is repeated below for convenience.

\[
C = \begin{bmatrix}
0.0865 & 0.0222 & 0.0272 & 0 & 0 & 0 \\
0.0865 & 0.0272 & 1.0186 & 0 & 0 & 0 \\
& & & 0.1206 & 0 & 0 \\
& & & & 0.1206 & 0 \\
& & & & & 0.0322 \\
\end{bmatrix} \text{ GPa,}
\]

\[
\rho = 64 \text{ kg/m}^3
\]

**Summary and discussion**

**Summary**

This manuscript discussed three different methods to retrieve the ultrasonic GW dispersion curves. First, a theoretical recap for the three different methods (the UAM, SAFE method and DISPERSE) was given.

---

\(x\) [1/m]

\(x\) [1/m]

\(x\) [1/m]

\(x\) [1/m]
For multi-layered media, the UAM formulation utilized the TMM assemble the final matrix to retrieve the dispersion curves. In DISPERSE, on the other hand, the GMM is applied to obtain an accurate result. The SAFE method uses a standard FEM assembly approach for multi-layered media.

The three different methods were utilized to retrieve the dispersion curves for four major cases:

(a) single layer media;
(b) multi-layer media;
(c) laminates;
(d) sandwich structure.

For the single layer and multi-layer media cases four different materials were examined:

(a) isotropic, aluminum;
(b) unidirectional;
(c) orthotropic;
(d) monoclinic material.

For the laminate case, three conventional laminates and one fiber metal laminate (a combination of isotropic/anisotropic media) were investigated. As last, a sandwich structure was evaluated to investigate the effect of structures with a large thickness on the algorithms.

Figure 4. Dispersion curves obtained the UAM, SAFE method and DISPERSE for a laminate: (a) two layered laminate [0/90]; (b) five layered laminate [0/45/90]; (c) eight layer quasi-isotropic laminate [±45/0/90], and eight layer fiber metal laminate [±45/Al/90].

Figure 5. Dispersion curves obtained with the UAM, SAFE method and DISPERSE for a sandwich laminate.
Due to the fact the SAFE method is a finite element method approach, its accuracy is dependent on the number of elements through the thickness. To obtain accurate results the number of elements was set to 20 such that a converged solution was obtained in each case.

Discussion

The UAM and SAFE method provided a complete set of accurate solutions for all the investigated cases. DISPERSE provided accurate results for the single layer media case. For multi-layer media the results were acceptable except for the monoclinic sub-case. For a 10 layered monoclinic laminate the A0 wave mode and several higher order modes were not retrieved. For laminates and the sandwich the same problem as for multi-layer media repeated: some missing higher order modes.

Based on the investigation presented in this manuscript it can be concluded that the UAM performed up to standard and DISPERSE had troubles when the propagation media consisted of multiple layers. Furthermore, it is interesting to highlight the performance of the UAM using TMM, despite the general trend present in the literature, the TMM did not show signs of numeric instabilities for high wave-number, high phase velocity combinations. On the contrary, in several cases it out performed DISPERSE which is based on the GMM.

Nevertheless, the authors want to state that DISPERSE is a powerful software package and it has its own common practice with respect to modeling: two or more adjacent identical layers should be modeled as one layer with an equivalent thickness. In addition, an increase in understanding of the options available within DISPERSE will lead to satisfactory results for any case.

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Note

1. It is important that indices 1, 2 and 3 in equation (11) need to interchange for x, y and z to maintain the DISPERSE notation format.

References


