Fast and Accurate Analytical Model to Solve Inverse Problem in SHM using Lamb Wave Propagation

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ABSTRACT

Lamb wave propagation is at the center of attention of researchers for structural health monitoring of thin walled structures. This is due to the fact that Lamb wave modes are natural modes of wave propagation in these structures with long travel distances and without much attenuation. This brings the prospect of monitoring large structure with few sensors/actuators. However the problem of damage detection and identification is an ”inverse problem” where we do not have the luxury to know the exact mathematical model of the system. On top of that the problem is more challenging due to the confounding factors of statistical variation of the material and geometric properties. Typically this problem may also be ill posed. Due to all these complexities the direct solution of the problem of damage detection and identification in SHM is impossible. Therefore an indirect method using the solution of the “forward problem” is popular for solving the “inverse problem”. This requires a fast forward problem solver. Due to the complexities involved with the forward problem of scattering of Lamb waves from damages researchers rely primarily on numerical techniques such as FEM, BEM, etc. But these methods are slow and practically impossible to be used in structural health monitoring. We have developed a fast and accurate analytical forward problem solver for this purpose. This solver, CMEP (complex modes expansion and vector projection), can simulate scattering of Lamb waves from all types of damages in thin walled structures fast and accurately to assist the inverse problem solver.

Keywords: Structural Health Monitoring, Inverse Problem, Scattering, Lamb Wave, Analytical Model

1 INTRODUCTION

Lamb waves are capable of inspecting large structures for non-destructive evaluation (NDE) and structural health monitoring (SHM) because they travel long distances without much attenuation. Therefore interaction of Lamb waves with damage has been an important topic of research. However, the detection and characterization of damage using Lamb wave propagation involves a difficult inverse problem which requires fast and accurate prediction of the scattered waves. But the prediction of the scattered waves is highly challenging because of the existence of multiple dispersive modes of Lamb waves at any frequency along with mode conversion at the damage location. Therefore, well-developed numerical methods such as the finite element method (FEM) and the boundary element method (BEM) have been popular [1–4]. However commercial FEM codes are time consuming and they do not provide much insight of the wave field in the structure, especially near the damage location. Therefore, for efficiency of simulation, researchers have developed hybrid methods using FEM, BEM, and normal mode expansion [2,4–6]. For interaction with horizontal cracks, Karim et al. [7] and Gunawan et al. [5] combined FEM with analytical approach for faster prediction of the scatter field of Lamb waves. Full analytical approaches such as Wiener-Hopf technique and higher order plate theories were also used by Rocklin [8] and Wang et al. [9] to solve this problem. Glushkov et al. have proposed the layered element method (LEM) which, unlike BEM, satisfies the plate boundary conditions by formulation [10]. They also proposed a simplified analytical model based on Kirchhoff plate theory for fast simulation for corrosion type damages [11]. However, the prospect of developing full analytical models based on normal mode expansion has also attracted attention for possible speed and accuracy [12–18].

One of the main challenges of such a method is to satisfy the thickness dependent boundary conditions [6,15]. Gregory et al. [14,19] developed the ‘projection method’ to satisfy these continuous boundary conditions. This method was developed to predict the singularity in stresses in the case of geometric discontinuities like a vertical crack [16]. A scalar form of the projection method was also used by Grahn [13]; though simple, this proved not to be very stable and to have slow convergence due to the use of simple sine and cosine functions [18]. Moreau et al. [17]

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used the displacement components of the complex Lamb wave modes instead of simple sine and cosine functions and attained a faster convergence in the projection method.

In this paper, we present an efficient analytical method for the prediction of Lamb wave scatter field using complex mode expansion and vector projection. We improved in several ways upon previous authors. One improvement is that we modified the projection method to take advantage of the power flow associated with Lamb wave modes. As different from the references [13], [17], the stress boundary conditions are projected onto the conjugate of the displacement modeshapes and the displacement boundary conditions are projected onto the conjugate of the stress modeshapes. Therefore this method transforms the stress and displacement equations into power equations. Another improvement is that we applied the projection to all the unknown wave fields. Our approach leads to fast convergence in terms of the number of complex modes needed in the modal expansion because it creates dominant diagonal terms in the matrix equation. It also ensures that, when the method has converged, the power balance is automatically satisfied. We call this method complex modes expansion with vector projection (CMEP). It is a Galerkin type approach where we implemented vector projection of the boundary conditions directly using the power flow expression. We illustrate the CMEP method by applying it to cases of corrosion (notch), a vertical crack, and a horizontal crack (disbond or delamination).

2 INVERSE PROBLEM OF STRUCTURAL HEALTH MONITORING

The goal of a structural health monitoring system and a nondestructive evaluation method is to determine whether a structure is damaged or not. The aim is to use transducers to actuate and sense Lamb waves in a plate structure; the detection and characterization of damage will be done based on the scattered wave signals. However, this inverse problem, like other diagnostic task, can show some very basic difficulties. From mathematical point of view an inverse problem is called ill-posed if at least one of the following three conditions is not satisfied:

1. The inverse problem has a solution
2. This solution is unique
3. Small disturbances in data causes only small deviations in the solution of the inverse problem

In the case of damage detection and identification, we know only the diagnostic input signal to the structure and the received output signal from the structure. The structural system and its parameters remains a black box where we don’t know if the system is linear or non-linear or what type of non-linearity describes the system. Therefore, in the case of structural health monitoring or nondestructive evaluation, the three criteria above cannot be guaranteed [20]. Therefore, the direct solution of inverse problem is impossible or at best inconclusive. Therefore, the most common approach to solve the inverse problem involves a forward problem solver (predictor) which can predict the received signal for various cases of damage and statistical variations of the structural properties. Also, for structural health monitoring the solution process has to be in real time or within a very short time after sensing the scattered waves. However, the numerical techniques, e.g. FEM, BEM, etc., that are commonly used as the forward problem solver are slow and cannot perform this task in a short time. Therefore, a much faster predictor is needed; an analytical predictor is desire for the speed, accuracy, and reliability that are needed.

3 FAST ANALYTICAL PREDICTOR: (CMEP)

3.1 Complex Solution of Rayleigh-Lamb Equation

Lamb waves are plate guided waves with multiple propagating modes at any frequency of excitation. The characteristic equation of Lamb waves is known as the Rayleigh-Lamb Equation [21] and is expressed in terms of the dimensionless wavenumber \( K = \xi d \) and dimensionless frequency \( \Omega = \omega d / c_s \) as

\[
\begin{align*}
\xi_s (K, \Omega) &= \left(K^2 - a^2\right)^2 \sin a \cos b + 4abK^2 \cos a \sin b = 0 \quad \text{(Symmetric)} \\
\xi_s (K, \Omega) &= \left(K^2 - a^2\right)^2 \sin b \cos a + 4abK^2 \cos b \sin a = 0 \quad \text{(Antisymmetric)}
\end{align*}
\]

where, \( a = \left(\Omega^2 - K^2\right)^{1/2} \), \( b = \left(\Omega^2 / \kappa^2 - K^2\right)^{1/2} \), and \( d \) is the half-thickness of the plate. The parameter \( \kappa = c_p / c_s \) and \( c_p \), \( c_s \) are the pressure and shear wave speeds respectively. The wave motion associated with each root \( K \) for given \( \Omega \) is expressed as

\[
F(X, Y, T) = A(Y) e^{-K_{real}X} e^{iK_{imag}X} e^{i\Omega T}; \quad K = K_{real} + iK_{imag}
\]
where, \( X = x/d \), \( Y = y/d \) are dimensionless propagation direction and thickness direction respectively and \( T = tc/d \) being dimensionless time. At any frequency \( \Omega \), Equation (1) has an infinite number of roots \( K, -\bar{K}, -K \), and \( \bar{K} \) in all four quadrants of the complex plane; a finite number of these roots are real (Figure 1 (a)), a finite number of them are imaginary (Figure 1 (b)), and the remainder are complex roots (Figure 1 (c)). It is apparent from Equation (2) that the wave motions associated with these roots are harmonic in both time and space for \( K \) being real, harmonic in time and exponentially decaying amplitude in space for \( K \) being imaginary, and harmonic in both time and space with exponentially decaying amplitude for \( K \) being complex. The positive real roots are assumed to be associated with modes propagating in the positive \( X \) direction. Also, for the wave motion to be physically plausible, the positive imaginary roots and the complex roots with positive imaginary parts are assumed to be associated with the positive \( X \) direction as they decay exponentially in the positive \( X \) direction. It is well known that the scattering of Lamb waves from geometric discontinuities involves all these wave modes in order to satisfy the boundary conditions [22,23]. We have developed an efficient and accurate algorithm to find these roots iteratively and sorted the complex roots with ascending imaginary part.

### 3.2 CMEP for Corrosion

To begin our analysis, let us consider a plate with a cross section as shown Fig. 2. We assume that there is an incident straight-crested Lamb wave mode travelling from the left towards a notch created by surface corrosion. Upon interacting with the notch, it will result in reflected wave modes, transmitted wave modes, and wave modes trapped in the notch. As shown in Fig. 2, the notch is located at a distance \( x = x_0 \) with the thickness of the plate being \( h_1 \). At the notch the plate has been corroded to a thickness \( h_2 \) with the width of the notch being \( L = 2b \). At the notch, we define depth ration as \( R_t = (h_1 - h_2)/h_1 \) and width ratio as \( R_w = 2b/h_1 \). Also, let us imagine that the incident wave field is represented by \( (\Phi_0, H_0) \), travelling in +ve \( x \) direction in the Region 1. We define the reflected wave field as \( (\Phi_1, H_1) \) and the transmitted wave field in Region 2 as \( (\Phi_2, H_2) \). We also define the trapped wave field inside the notch in Region 2 as \( (\Phi_3, H_3) \). The incident and scattered wave fields satisfy the generic wave equations

\[
\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = \frac{1}{C_p^2} \Phi
\]

\[
\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = \frac{1}{C_s^2} H
\]

where, \( C_p \) and \( C_s \) are the wave speeds of pressure wave and shear waves, respectively. These must satisfy the zero-stress boundary condition at the top and bottom of the plate,

\[
\sigma_{yy} \begin{cases} 
\left( x < x_0 - b; \ y = \pm \frac{h_1}{2} \right) & \left( x > x_0 + b; \ y = \pm \frac{h_1}{2} \right) \\
\left( x = x_0 + b; \ y = \pm \frac{h_1}{2} \right) & \left( x = x_0 - b; \ y = \pm \frac{h_1}{2} \right)
\end{cases} = 0
\]

\[
\tau_{xy} \begin{cases} 
\left( x < x_0 - b; \ y = \pm \frac{h_1}{2} \right) & \left( x > x_0 + b; \ y = \pm \frac{h_1}{2} \right) \\
\left( x = x_0 + b; \ y = \pm \frac{h_1}{2} \right) & \left( x = x_0 - b; \ y = \pm \frac{h_1}{2} \right)
\end{cases} = 0
\]

The stress and displacement fields associated with the wave fields are expressed as

\[
\sigma = \begin{bmatrix} \sigma_{xx} & \tau_{xx} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix}; \quad \tilde{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \tau_{xx} \end{bmatrix}; \quad \tilde{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix}
\]

where, \( \sigma \) is stress tensor, \( \tilde{\sigma} \) is stress vector, and \( \tilde{u} \) is displacement vector. For the displacements, subscripts \( x \) and \( y \) indicate the directions of the displacement. For stresses, the subscripts \( xx \), \( yy \) stand for the normal stress in \( x \), \( y \) directions, respectively and the subscript \( xy \) stands for the shear stress. The boundary conditions at the notch are illustrated in Fig. 3(a). The boundary conditions at the interface at \( x = x_0 - b \) are,
\[
(\sigma_0 + \sigma_1) \cdot \hat{n}_x = \begin{cases} 
0, & x = x_0 - b, \\ \sigma_2 \cdot \hat{n}_x, & x = x_0 - b, \quad -h_b/2 \leq y \leq h_b/2 
\end{cases}
\]

(6)

\[
\bar{u}_0 + \bar{u}_1 = \bar{u}_2,
\]

(7)

The boundary conditions at the interface at \( x = x_0 + b \) are,

\[
\sigma_3 \cdot \hat{n}_x = \begin{cases} 
0, & x = x_3(y), \quad h_b/2 \leq y \leq h_b/2 \\
\sigma_2 \cdot \hat{n}_x, & x = x_3(y), \quad h_b/2 \leq y \leq h_b/2 
\end{cases}
\]

(8)

\[
\hat{u}_3 = \hat{u}_2, \quad x = x_0 + b, \quad -h_b/2 \leq y \leq h_b/2
\]

(9)

where \( \hat{n}_x \) are the unit surface normal vectors of the interface surfaces represented by \( x_i(y), x_j(y) \) and \( \hat{n}_x \) is the unit vector in +ve \( x \) direction as shown in Fig. 3(a). Also note that for \( -h_b/2 \leq y \leq h_b/2 \) \( \hat{n}_x = \hat{n}_x \) and \( \hat{n}_x = \hat{n}_x \). The subscript 0 stands for incident waves in Region 1, subscript 1 stands for reflected waves in Region 1, subscript 2 stands for the trapped waves in Region 2 and subscript 3 stands for transmitted waves in Region 3. Let us assume the incident wave field to be a harmonic wave field of the form

\[
\Phi_0 = f_0(y) e^{i \omega y}, \quad H_0 = i h_0(y) e^{i \omega y}
\]

(10)

Assuming \( \xi_0 \) to be one of the roots of Rayleigh-Lamb equation for the plate in Region 1, Eq. (3) and (4) are satisfied by definition of Lamb waves and the incident wave becomes one of the modes of Lamb waves. The time dependent part \( e^{i \alpha t} \) is ignored assuming linearity of the problem.

### 3.2.1 Complex Modes Expansion of the Scattered Wave Field

In this section, we present the general concept of our CMEP algorithm. We assume that the transmitted, reflected, and trapped wave fields to have harmonic expressions similar to the incident wave field \( \Phi_0, H_0 \) given by Eq. (10). Since the boundary conditions at the notch cannot be satisfied by assuming any finite number of Lamb wave modes [12], we expand these transmitted, reflected, and trapped wave fields in terms of all possible complex Lamb wave modes corresponding to the complex roots of Rayleigh-Lamb frequency equation. Therefore, the scattered wave fields are expressed as

\[
\Phi_1 = \sum_{n_1=1}^{\infty} C_{1B,n_1} \Phi_{1B,n_1} = \sum_{n_1=1}^{\infty} C_{1B,n_1} f_{n_1}(y_1) e^{i \xi_{B,n_1} y_1}; H_1 = \sum_{n_1=1}^{\infty} C_{1B,n_1} H_{1B,n_1} = \sum_{n_1=1}^{\infty} C_{1B,n_1} h_{n_1}(y_1) e^{i \xi_{B,n_1} y_1}
\]

\[
\Phi_2 = \sum_{n_2=1}^{\infty} (C_{2F,n_2} \Phi_{2F,n_2} + C_{2B,n_2} \Phi_{2B,n_2}) = \sum_{n_2=1}^{\infty} \left[ C_{2F,n_1} f_{n_2}(y_2) e^{i \xi_{F,n_2} y_2} + C_{2B,n_2} f_{n_2}(y_2) e^{i \xi_{B,n_2} y_2} \right]
\]

\[
H_2 = \sum_{n_2=1}^{\infty} (C_{2F,n_2} H_{2F,n_2} + C_{2B,n_2} H_{2B,n_2}) = \sum_{n_2=1}^{\infty} \left[ C_{2F,n_1} i h_{n_2}(y_2) e^{i \xi_{F,n_2} y_2} + C_{2B,n_2} i h_{n_2}(y_2) e^{i \xi_{B,n_2} y_2} \right]
\]

\[
\Phi_3 = \sum_{n_3=1}^{\infty} C_{3F,n_3} \Phi_{3F,n_3} = \sum_{n_3=1}^{\infty} C_{3F,n_3} f_{n_3}(y_3) e^{i \xi_{F,n_3} y_3}; H_3 = \sum_{n_3=1}^{\infty} C_{3F,n_3} H_{3F,n_3} = \sum_{n_3=1}^{\infty} C_{3F,n_3} i h_{n_3}(y_3) e^{i \xi_{F,n_3} y_3}
\]

(11)

where \( y_1 \) and \( y_2 \) are connected by the expression \( y_2 = y_1 + a \) with \( a = (h_1 - h_2)/2 \) being the eccentricity between Region 1 and Region 2 and \( y_1 = y_3 \). The wavenumber \( \xi_{1B,n_1} \) is the \( n_1 \)th complex root of the Rayleigh-Lamb equation corresponding to backward propagating Lamb waves in Region 1 and the wavenumbers \( \xi_{2F,n_2} \) and \( \xi_{2B,n_2} \) are \( n_2 \)th complex root of the Rayleigh-Lamb equation corresponding to forward and backward propagating waves in Region 2, respectively. Similarly, wavenumber \( \xi_{3F,n_3} \) is the \( n_3 \)th complex root of the Rayleigh-Lamb
equation corresponding to forward propagating waves in Region 3. The coefficient $C_{1B,n_1}$ is the unknown amplitude of the $n_1$ th mode of backward propagating Lamb waves in Region 1 whereas $C_{2F,n_2}$ and $C_{2B,n_2}$ are the unknown amplitudes of the $n_2$ th mode of forward and backward propagating Lamb waves in Region 2, respectively. Similarly, the coefficient $C_{3F,n_3}$ is the unknown amplitude of the $n_3$ th mode of forward propagating Lamb waves in Region 3. Eq. (11) expresses the scattered wave field as the summation of all possible complex Lamb wave modes at a given frequency. The amplitudes $C_{1B,n_1}$, $C_{2F,n_2}$, $C_{2B,n_2}$ and $C_{3F,n_3}$ of these modes have to be determined through the boundary matching process. Recall the boundary conditions at the interfaces at $x = x_0 - b$ and $x = x_0 + b$ are given by Eq. (6), (7), (8), and (9). Also, note that the plate boundary conditions given by Eq. (3) and (4) are satisfied by the definition of Lamb waves. We express the stress and displacement fields of Eq. (6), (7), (8), and (9) using the complex Lamb wave mode expansion of Eq. (11), i.e.,

\[
\mathbf{u}_1 = \sum_{n_1=1}^{\infty} C_{1B,n_1} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{1B,n_1}; \quad \mathbf{\sigma}_1 = \sum_{n_1=1}^{\infty} C_{1B,n_1} \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{1B,n_1}
\]

\[
\mathbf{u}_2 = \sum_{n_2=1}^{\infty} \left( C_{2F,n_2} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{2F,n_2} + C_{2B,n_2} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{2B,n_2} \right)
\]

\[
\mathbf{\sigma}_2 = \sum_{n_2=1}^{\infty} \left( C_{2F,n_2} \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{2F,n_2} + C_{2B,n_2} \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{2B,n_2} \right)
\]

\[
\mathbf{u}_3 = \sum_{n_3=1}^{\infty} C_{3F,n_3} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{3F,n_3}; \quad \mathbf{\sigma}_3 = \sum_{n_3=1}^{\infty} C_{3F,n_3} \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{3F,n_3}
\]

In the same vein, the incident wave field uses subscript 0, i.e.,

\[
\mathbf{u}_0 = \begin{bmatrix} u_x \\ u_y \end{bmatrix}_0; \quad \mathbf{\sigma}_0 = \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_0
\]

Using Eq. (12), (13) into Eq. (6), (7), (8) and (9) yields

\[
(\mathbf{\sigma}_0 + \mathbf{\sigma}_1) \cdot \hat{n}_{x_1} = 0, \quad x_1(y), \quad h_2 - h_1/2 \leq y \leq h_1/2
\]

(14) 

\[
(\mathbf{\sigma}_0 + \mathbf{\sigma}_1) \cdot \hat{n}_{x_2} = \mathbf{\sigma}_2, \quad x = x_0 - b, \quad -h_1/2 \leq y \leq h_2 - h_1/2
\]

(15) 

\[
(\mathbf{\sigma}_0 + \mathbf{\sigma}_1) \cdot \hat{n}_{x_3} = 0, \quad x_3(y), \quad h_2 - h_1/2 \leq y \leq h_1/2
\]

(16) 

\[
(\mathbf{\sigma}_0 + \mathbf{\sigma}_1) \cdot \hat{n}_{x_4} = \mathbf{\sigma}_3, \quad x = x_0 + b, \quad -h_1/2 \leq y \leq h_2 + h_1/2
\]

(17) 

Therefore, Eq. (14), (15), (16), and (17) represent the thickness dependent boundary conditions at the notch.

### 3.2.2 Vector Projection of the Boundary Conditions

To make Eq. (14), (15), (16), and (17) independent of $y$, we follow Grahn [13] and project them onto appropriate complete orthogonal vector spaces as described. But, different from [13], we do not use generic sine and cosine functions, instead, we use the time averaged power flow expression which uses stress-velocity product [24]. Thus, in Region 1, we project the stress boundary conditions onto the conjugate displacement vector space of the complex Lamb wave modes; in Region 2, we project the displacement boundary conditions onto the conjugate stress vector space of the complex Lamb wave modes. By the same token, in Region 3, we project the stress boundary conditions onto the conjugate displacement vector space of the complex Lamb wave modes. By doing so, the CMEP
formulation automatically incorporates the average power flow associated with the reflected, transmitted, and trapped wave fields. This approach has two main advantages: first, following the time averages power flow associated with the modes, it creates dominant diagonal terms in the final matrix equation which leads to fast convergence; second, it transforms the stress and displacement equations at the interface into the equations representing the power flow balance across the interface between the two separate wave fields. In short, this approach incorporates the balance of power flow across the interface achieving fast convergence.

The projection vector space for Eq. (14) is

$$\vec{u}_{1B} = \text{conj} \left[ \begin{array}{c} u_x \\ u_y \end{array} \right]_{1B, \tilde{n}_1} = \left[ \begin{array}{c} \vec{u}_x \\ \vec{u}_y \end{array} \right]_{1B, \tilde{n}_1}, \quad \tilde{n}_1 = 1, 2, 3, \ldots$$  \hspace{1cm} (18)

After projecting Eq. (14) onto Eq. (18), the stress boundary conditions in Eq. (14) take the form,

$$\int_{h_2-h_1/2}^{h_2} (\sigma_0 + \sigma_1) \cdot \hat{n}_1 \cdot \vec{u}_{1B}dy + \int_{h_1/2}^{h_2-h_1/2} (\bar{\sigma}_0 + \bar{\sigma}_1) \cdot \vec{u}_{1B}dy = \int_{h_1/2}^{h_2-h_1/2} \bar{\sigma}_2 \cdot \vec{u}_{1B}dy$$

$$\Rightarrow \sum_{n_1=1}^{\infty} C_{2F,n_1} \left( \begin{array}{c} \sigma_{xx} \\ \tau_{xy} \end{array} \right)_{2F, n_1} \cdot \left[ \begin{array}{c} \vec{u}_x \\ \vec{u}_y \end{array} \right]_{1B, \tilde{n}_1} \bigg|_{h_2-h_1/2}^{h_2-h_1/2} + C_{2B,n_2} \left( \begin{array}{c} \sigma_{xx} \\ \tau_{xy} \end{array} \right)_{2B, n_2} \cdot \left[ \begin{array}{c} \vec{u}_x \\ \vec{u}_y \end{array} \right]_{1B, \tilde{n}_1} \bigg|_{h_2-h_1/2}^{h_2-h_1/2}$$

$$= \sum_{n_1=1}^{\infty} C_{1B,n_1} \left( \begin{array}{c} \sigma_{xx} \\ \tau_{xy} \end{array} \right)_{1B, n_1} \cdot \hat{n}_1 \cdot \left[ \begin{array}{c} \vec{u}_x \\ \vec{u}_y \end{array} \right]_{1B, \tilde{n}_1} \bigg|_{h_2-h_1/2}^{h_2-h_1/2} + \sum_{n_2=1}^{\infty} C_{1B,n_2} \left( \begin{array}{c} \sigma_{xx} \\ \tau_{xy} \end{array} \right)_{1B, n_2} \cdot \left[ \begin{array}{c} \vec{u}_x \\ \vec{u}_y \end{array} \right]_{1B, \tilde{n}_1} \bigg|_{h_2-h_1/2}^{h_2-h_1/2}$$

$$\Rightarrow \left( \begin{array}{c} \sigma_{xx} \\ \tau_{xy} \end{array} \right)_{1B, n_1} \cdot \hat{n}_1 \cdot \left[ \begin{array}{c} \vec{u}_x \\ \vec{u}_y \end{array} \right]_{1B, \tilde{n}_1} \bigg|_{h_2-h_1/2}^{h_2-h_1/2} + \sum_{n_2=1}^{\infty} C_{1B,n_2} \left( \begin{array}{c} \sigma_{xx} \\ \tau_{xy} \end{array} \right)_{1B, n_2} \cdot \left[ \begin{array}{c} \vec{u}_x \\ \vec{u}_y \end{array} \right]_{1B, \tilde{n}_1} \bigg|_{h_2-h_1/2}^{h_2-h_1/2}$$

where, $\int_{a}^{b} P \cdot Q dy = \langle P, Q \rangle_{a}^{b}$ represents the inner product. Similarly, the projection vector space for Eq. (16) is

$$\vec{u}_{3F} = \text{conj} \left[ \begin{array}{c} u_x \\ u_y \end{array} \right]_{3F, \tilde{n}_3} = \left[ \begin{array}{c} \vec{u}_x \\ \vec{u}_y \end{array} \right]_{3F, \tilde{n}_3}, \quad \tilde{n}_3 = 1, 2, 3, \ldots$$ \hspace{1cm} (20)

Upon projecting Eq. (16) onto Eq. (20), the stress boundary conditions in Eq. (16) take the form,

$$\int_{h_1/2}^{h_2-h_1/2} \sigma_3 \cdot \vec{u}_{3F}dy + \int_{h_1/2}^{h_2-h_1/2} \bar{\sigma}_3 \cdot \vec{u}_{3F}dy = \int_{h_1/2}^{h_2-h_1/2} \bar{\sigma}_2 \cdot \vec{u}_{3F}dy$$

$$\Rightarrow \sum_{n_2=1}^{\infty} C_{2F,n_2} \left( \begin{array}{c} \sigma_{xx} \\ \tau_{xy} \end{array} \right)_{2F, n_2} \cdot \left[ \begin{array}{c} \vec{u}_x \\ \vec{u}_y \end{array} \right]_{3F, \tilde{n}_3} \bigg|_{h_2-h_1/2}^{h_2-h_1/2} + C_{2B,n_3} \left( \begin{array}{c} \sigma_{xx} \\ \tau_{xy} \end{array} \right)_{2B, n_3} \cdot \left[ \begin{array}{c} \vec{u}_x \\ \vec{u}_y \end{array} \right]_{3F, \tilde{n}_3} \bigg|_{h_2-h_1/2}^{h_2-h_1/2}$$

$$= \sum_{n_2=1}^{\infty} C_{3F,n_2} \left( \begin{array}{c} \sigma_{xx} \\ \tau_{xy} \end{array} \right)_{3F, n_2} \cdot \hat{n}_3 \cdot \left[ \begin{array}{c} \vec{u}_x \\ \vec{u}_y \end{array} \right]_{3F, \tilde{n}_3} \bigg|_{h_2-h_1/2}^{h_2-h_1/2} + \sum_{n_3=1}^{\infty} C_{3F,n_3} \left( \begin{array}{c} \sigma_{xx} \\ \tau_{xy} \end{array} \right)_{3F, n_3} \cdot \left[ \begin{array}{c} \vec{u}_x \\ \vec{u}_y \end{array} \right]_{3F, \tilde{n}_3} \bigg|_{h_2-h_1/2}^{h_2-h_1/2}$$

$$n_2, n_3 = 1, 2, 3, \ldots$$

The projection vector space for Eq. (15) is,

$$\bar{\sigma}_{2F} = \text{conj} \left[ \begin{array}{c} \sigma_{xx} \\ \tau_{xy} \end{array} \right]_{2F, \tilde{n}_2} = \left[ \begin{array}{c} \bar{\sigma}_x \\ \bar{\sigma}_y \end{array} \right]_{2F, \tilde{n}_2}, \quad \tilde{n}_2 = 1, 2, 3, \ldots$$ \hspace{1cm} (22)

After projecting Eq. (15) onto Eq. (22), the displacement boundary conditions in Eq. (15), take the form,
\[
\int_{-h/2}^{h_{-h/2}} (\vec{u}_0 + \vec{u}_1) \cdot \vec{\sigma}_{2b} \, dy = \int_{-h/2}^{h_{-h/2}} \vec{u}_2 \cdot \vec{\sigma}_{2b} \, dy
\]

\[
\Rightarrow \sum_{n_l=1}^{\infty} \left( C_{2,F,n_l} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{2,F,n_l} \begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \end{bmatrix}_{2,F,n_l} \right)_{-h/2}^{h_{-h/2}} + C_{2,B,n_l} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{2,B,n_l} \begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \end{bmatrix}_{2,B,n_l} \right)_{-h/2}^{h_{-h/2}} - \sum_{n_l=1}^{\infty} C_{1,B,n_l} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{1,B,n_l} \begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \end{bmatrix}_{1,B,n_l} \right)_{-h/2}^{h_{-h/2}} = \left( \begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \end{bmatrix}_{2,F,n_l} \right)_{-h/2}^{h_{-h/2}} n_1, n_2 = 1, 2, 3, ...
\]

(23)

The projection vector space for Eq. (17) is,

\[
\vec{\sigma}_{2b} = \text{conj} \left[ \begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \end{bmatrix}_{2,F,n_l} \right], n_2 = 1, 2, 3, ...
\]

(24)

Upon projecting Eq. (17) onto Eq. (24), the displacement boundary conditions in Eq. (17) take the form,

\[
\int_{-h/2}^{h_{-h/2}} \vec{u}_3 \cdot \vec{\sigma}_{2b} \, dy = \int_{-h/2}^{h_{-h/2}} \vec{u}_2 \cdot \vec{\sigma}_{2b} \, dy
\]

\[
\Rightarrow \sum_{n_l=1}^{\infty} \left( C_{2,F,n_l} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{2,F,n_l} \begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \end{bmatrix}_{2,F,n_l} \right)_{-h/2}^{h_{-h/2}} + C_{2,B,n_l} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{2,B,n_l} \begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \end{bmatrix}_{2,B,n_l} \right)_{-h/2}^{h_{-h/2}} = \sum_{n_l=1}^{\infty} C_{3,F,n_l} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{3,F,n_l} \begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \end{bmatrix}_{3,F,n_l} \right)_{-h/2}^{h_{-h/2}} n_2, n_3 = 1, 2, 3, ...
\]

(25)

### 3.2.3 Numerical Solution

For numerical calculation we consider finite values for the indices \( n_1, n_2, n_3, \bar{n}_1, \bar{n}_2, \bar{n}_3 \). We assume, \( n_1 = 1, 2, 3, ..., N_1 \), \( n_2 = 1, 2, 3, ..., N_2 \), \( n_3 = 1, 2, 3, ..., N_3 \), \( \bar{n}_1 = 1, 2, 3, ..., \bar{N}_1 \), \( \bar{n}_2 = 1, 2, 3, ..., \bar{N}_2 \), \( \bar{n}_3 = 1, 2, 3, ..., \bar{N}_3 \). Then, Eq. (19) contains \( \bar{N}_1 \) linear equations with \( \left(N_1 + 2N_2\right) \) unknowns and Eq. (21) contains \( \bar{N}_2 \) linear equations with \( \left(N_1 + 2N_2\right) \) unknowns. Also Eq. (23) contains \( \bar{N}_2 \) linear equations with \( \left(N_1 + 2N_2\right) \) unknowns. Recall that the \( \left(N_1 + 2N_2 + N_3\right) \) unknowns are the complex Lamb wave mode amplitudes \( \left(C_{1B,n_1}, C_{2F,n_2}, C_{2B,n_2}, C_{3F,n_3}\right) \). Thus, Eq. (19), (21), (23), (25) combined are a set of \( \left(N_1 + 2N_2 + \bar{N}_3\right) \) linear algebraic equations in \( \left(N_1 + 2N_2 + N_3\right) \) unknowns \( C_{1B,n_1}, C_{2F,n_2}, C_{2B,n_2}, \) and \( C_{3F,n_3} \). By assuming \( N_1 = N_2 = N_3 = \bar{N}_1 = \bar{N}_2 = N \), we get \( 4N \) equations in \( 4N \) unknowns. Then Eq. (19) and (21) can be written as

\[
\begin{bmatrix} A \end{bmatrix}_{N \times N} \left\{ C_{2F} \right\}_{N \times 1} + \begin{bmatrix} B \end{bmatrix}_{N \times N} \left\{ C_{2B} \right\}_{N \times 1} - \begin{bmatrix} D \end{bmatrix}_{N \times N} \left\{ C_{1B} \right\}_{N \times 1} - \begin{bmatrix} E \end{bmatrix}_{N \times N} \left\{ C_{3F} \right\}_{N \times 1} = \begin{bmatrix} 0 \end{bmatrix}_{N \times 1}
\]

(26)

\[
\begin{bmatrix} F \end{bmatrix}_{N \times N} \left\{ C_{2F} \right\}_{N \times 1} + \begin{bmatrix} G \end{bmatrix}_{N \times N} \left\{ C_{2B} \right\}_{N \times 1} - \begin{bmatrix} H \end{bmatrix}_{N \times N} \left\{ C_{1B} \right\}_{N \times 1} - \begin{bmatrix} H \end{bmatrix}_{N \times N} \left\{ C_{3F} \right\}_{N \times 1} = \begin{bmatrix} 0 \end{bmatrix}_{N \times 1}
\]

(27)

Similarly from Eq. (23) and (25) can be written as

\[
\begin{bmatrix} J \end{bmatrix}_{N \times N} \left\{ C_{2F} \right\}_{N \times 1} + \begin{bmatrix} K \end{bmatrix}_{N \times N} \left\{ C_{2B} \right\}_{N \times 1} - \begin{bmatrix} L \end{bmatrix}_{N \times N} \left\{ C_{1B} \right\}_{N \times 1} - \begin{bmatrix} M \end{bmatrix}_{N \times N} \left\{ C_{3F} \right\}_{N \times 1} = \begin{bmatrix} 0 \end{bmatrix}_{N \times 1}
\]

(28)
\[
\begin{bmatrix}
N_{N\times N} \{C_{2F}\}_{N\times 4} + O_{N\times N} \{C_{2B}\}_{N\times 4} - 0_{N\times N} \{C_{1B}\}_{N\times 4} - P_{N\times N} \{C_{2F}\}_{N\times 4} = 0_{N\times 4}
\end{bmatrix}
\]  

In Eq. (26), (27), (28) and (29) the coefficient matrices \([A], [B], [D], [E], [F], [G], [H], [J], [K], [L], [M], [N], [O] and [P]\) are known matrices containing the vector projected boundary conditions. Combining them we get

\[
\begin{bmatrix}
A & B & -D & 0 \\
F & G & 0 & -H \\
J & K & -L & 0 \\
N & O & 0 & -P
\end{bmatrix}
\begin{bmatrix}
C_{2F} \\
C_{2B} \\
C_{1B} \\
C_{3F}
\end{bmatrix}
= \begin{bmatrix}
E \\
0 \\
M \\
0
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
Q_{4N\times 4N} \{C\}_{4N\times 4N} = \{R\}_{4N\times 4N}
\end{bmatrix}
\]  

Eq. (30) can be solved for the unknown amplitudes of the reflected and transmitted Lamb wave modes as

\[
\{C\}_{4N\times 4N} = [Q]^{-1} [R]_{4N\times 4N}
\]  

### 3.3 CMEP for Crack

A crack can be assumed as a notch with vanishing width. There, to simulate the interaction of Lamb waves with a crack using CMEP, we have assumed \(b = 0\). Rest of the process is identical to the section 3.2.

### 3.4 CMEP for Disbond

To begin our analysis, let us consider a plate with a cross section as shown in Figure 4. We assume that there is an incident straight-crested Lamb wave mode travelling from the left towards a disbond in the form of a horizontal crack. Upon interacting with the disbond, it will result in reflected wave modes, transmitted wave modes, and wave modes trapped in the disbond. As shown in Figure 4, the disbond is located at a distance \(x = x_0\) in a plate with thickness \(h_1\). The disbond has a width \(L = 2b\) and is located at height \(h_2\) from the bottom of the plate. At the disbond, we define depth ration as \(R_y = (h_1 - h_2)/h_1\) and width ratio as \(R_w = 2b/h_1\). Also, let us imagine that the incident wave field is represented by \(\{\Phi_0, H_0\}\), travelling in +ve \(x\) direction in the Region 1. We define the reflected wave field as \(\{\Phi_1, H_1\}\) and the transmitted wave field in Region 4 as \(\{\Phi_4, H_4\}\). We also define the trapped wave field in the disbond area as \(\{\Phi_2, H_2\}\) and \(\{\Phi_3, H_3\}\) in Regions 2 and 3, respectively. The incident and scattered wave fields satisfy the generic wave equations (3). Equations (3) must satisfy the zero-stress boundary condition at the top and bottom of the plate,

\[
\begin{align*}
\sigma_{xy} &\bigg|_{x < x_0 - b; \ y = \pm \frac{h}{2}} = 0 \\
\tau_{xy} &\bigg|_{x < x_0 - b; \ y = \pm \frac{h}{2}} = 0 \\
\sigma_{yy} &\bigg|_{x_0 + b > x > x_0 - b; \ y = \frac{h}{2} - \frac{h}{2}} = 0 \\
\tau_{xy} &\bigg|_{x_0 + b > x > x_0 - b; \ y = \frac{h}{2} - \frac{h}{2}} = 0
\end{align*}
\]  

(Remaine...
\[
(\sigma_0 + \sigma_1) = \begin{cases}
\sigma_3, & x = x_0 - b, \quad h_2 - h_1/2 \leq y \leq h_1/2 \\
\sigma_2, & x = x_0 - b, \quad -h_1/2 \leq y \leq h_2 - h_1/2 \\
0, & \text{otherwise}
\end{cases}
\]

\[
\mathbf{u}_0 + \mathbf{u}_1 = \begin{cases}
\mathbf{u}_3, & x = x_0 - b, \quad h_2 - h_1/2 \leq y \leq h_1/2 \\
\mathbf{u}_2, & x = x_0 - b, \quad -h_1/2 \leq y \leq h_2 - h_1/2 \\
0, & \text{otherwise}
\end{cases}
\]

The boundary conditions at the interface at \( x = x_0 + b \) are,
\[
\begin{align*}
\mathbf{u}_4 &= \begin{cases}
\mathbf{u}_3, & x = x_0 + b, \quad h_2 - h_1/2 \leq y \leq h_1/2 \\
\mathbf{u}_2, & x = x_0 + b, \quad -h_1/2 \leq y \leq h_2 - h_1/2 \\
0, & \text{otherwise}
\end{cases} \\
\end{align*}
\]

The subscript 0 stands for incident waves in Region 1, subscript 1 stands for reflected waves in Region 1, subscripts 2 and 3 stand for the trapped waves in Regions 2 and 3, respectively. The subscript 4 stands for transmitted waves in Region 4. Let us assume the incident wave field in Region 1 to be harmonic of the form
\[
\Phi_0 = f_0(y) e^{i\omega x} \quad H_0 = i\theta_0(y) e^{i\omega x}
\]
where, \( \xi_0 \) satisfies the Rayleigh-Lamb equation for Region 1, and \( f_0(y), \theta_0(y) \) are the corresponding wave modes. The time dependent part \( e^{i\omega t} \) is ignored assuming linearity of the problem.

### 3.4.1 Complex Modes Expansion of the Scattered Wave Field

In this section, we present the general concept of our CMEP algorithm. We assume that the transmitted, reflected, and trapped wave fields to have harmonic expressions similar to the incident wave field \((\Phi_0, H_0)\) given by Equation (10). Since the boundary conditions at the horizontal crack cannot be satisfied by assuming any finite number of Lamb wave modes [12], we expand these transmitted, reflected, and trapped wave fields in terms of all possible complex Lamb wave modes corresponding to the complex roots of Rayleigh-Lamb frequency equation with unknown complex amplitudes. Therefore, the scattered wave fields are expressed as
\[
\begin{align*}
\Phi_1 &= \sum_{n_1=1}^{\infty} C_{B,n_1} \Phi_{1,B,n_1} = \sum_{n_1=1}^{\infty} C_{B,n_1} f_{n_1}(y_1) e^{i\xi_{B,n_1} x} + H_1 = \sum_{n_1=1}^{\infty} C_{B,n_1} H_{1,B,n_1} = \sum_{n_1=1}^{\infty} C_{B,n_1} i\theta_{n_1}(y_1) e^{i\xi_{B,n_1} x} \\
\Phi_2 &= \sum_{n_2=1}^{\infty} \left( C_{F,n_2} \Phi_{2,F,n_2} + C_{B,n_2} \Phi_{2,a,n_2} \right) = \sum_{n_2=1}^{\infty} \left( C_{F,n_2} f_{n_2}(y_2) e^{i\xi_{F,n_2} x} + C_{B,n_2} f_{n_2}(y_2) e^{i\xi_{B,a,n_2} x} \right) \\
H_2 &= \sum_{n_2=1}^{\infty} \left( C_{F,n_2} H_{2,F,n_2} + C_{B,n_2} H_{2,a,n_2} \right) = \sum_{n_2=1}^{\infty} \left( C_{F,n_2} i\theta_{n_2}(y_2) e^{i\xi_{F,n_2} x} + C_{B,n_2} i\theta_{n_2}(y_2) e^{i\xi_{B,a,n_2} x} \right) \\
\Phi_3 &= \sum_{n_3=1}^{\infty} \left( C_{3,F,n_3} \Phi_{3,F,n_3} + C_{3,B,n_3} \Phi_{3,a,n_3} \right) = \sum_{n_3=1}^{\infty} \left( C_{3,F,n_3} f_{n_3}(y_3) e^{i\xi_{3,F,n_3} x} + C_{3,B,n_3} f_{n_3}(y_3) e^{i\xi_{3,a,n_3} x} \right) \\
H_3 &= \sum_{n_3=1}^{\infty} \left( C_{3,F,n_3} H_{3,F,n_3} + C_{3,B,n_3} H_{3,a,n_3} \right) = \sum_{n_3=1}^{\infty} \left( C_{3,F,n_3} i\theta_{n_3}(y_3) e^{i\xi_{3,F,n_3} x} + C_{3,B,n_3} i\theta_{n_3}(y_3) e^{i\xi_{3,a,n_3} x} \right) \\
\Phi_4 &= \sum_{n_4=1}^{\infty} C_{4,F,n_4} \Phi_{4,a,n_4} = \sum_{n_4=1}^{\infty} C_{4,F,n_4} f_{n_4}(y_4) e^{i\xi_{4,F,n_4} x}; H_4 &= \sum_{n_4=1}^{\infty} C_{4,F,n_4} H_{4,a,n_4} = \sum_{n_4=1}^{\infty} C_{4,F,n_4} i\theta_{n_4}(y_4) e^{i\xi_{4,F,a,n_4} x}
\end{align*}
\]

where, subscripts \( B \) and \( F \) stand for backward and forward propagating wave. \( y_1, y_2, y_3 \) are connected by the expressions \( y_2 = y_1 + a_2 \) and \( y_3 = y_1 + a_3 \) with \( a_2 = (h_1 - h_2)/2 \) and \( a_3 = -h_2/2 \) being the eccentricities of
Using Equations(12), (13) into Equations (6), (7), (8) and (9) yields

\[
\bar{\mathbf{u}}_0 + \bar{\mathbf{\sigma}}_1 = \begin{cases} 
\bar{\mathbf{\sigma}}_3, & x = x_0 - b, \quad h_2 - h_1/2 \leq y \leq h_1/2 \\
\bar{\mathbf{\sigma}}_2, & x = x_0 - b, \quad -h_1/2 \leq y \leq h_2 - h_1/2 
\end{cases}
\]

\[
\bar{\mathbf{u}}_0 + \bar{\mathbf{u}}_1 = \bar{\mathbf{u}}_2, \quad x = x_0 - b, \quad h_2 - h_1/2 \leq y \leq h_1/2
\]

\[
\bar{\mathbf{u}}_0 + \bar{\mathbf{u}}_1 = \bar{\mathbf{u}}_2, \quad x = x_0 - b, \quad -h_1/2 \leq y \leq h_2 - h_1/2
\]
\[
\bar{\sigma}_4 = \begin{cases} 
\bar{\sigma}_3, & x = x_0 + b, \quad h_2 - h_1/2 \leq y \leq h_1/2 \\
\bar{\sigma}_2, & x = x_0 + b, \quad -h_1/2 \leq y \leq h_2 - h_1/2 
\end{cases}
\]

\[
\bar{u}_4 = \bar{u}_3, \quad x = x_0 + b, \quad h_2 - h_1/2 \leq y \leq h_1/2
\]

\[
\bar{u}_4 = \bar{u}_2, \quad x = x_0 + b, \quad -h_1/2 \leq y \leq h_2 - h_1/2
\]

Therefore, Equations (14), (15), (44), (16), (17) and (47) represent the thickness dependent boundary conditions at the horizontal crack.

### 3.4.2 Vector Projection of the Boundary Conditions

To make Equations (14), (15), (16), and (17) independent of \( y \), we follow Grahn [13] and project them onto appropriate complete orthogonal vector spaces. But, different from [13], we do not use generic sine and cosine functions, instead, we use the time averaged power flow expression which uses the stress-velocity product [24]. Thus, in Region 1, we project the stress boundary conditions (14), onto the conjugate displacement vector space of the complex Lamb wave modes; in Region 2 and 3, we project the displacement boundary conditions (15), (44), (17), and (47), onto the conjugate stress vector space of the complex Lamb wave modes. By the same token, in Region 4, we project the stress boundary conditions (16), onto the conjugate displacement vector space of the complex Lamb wave modes. By doing so, the CMEP formulation incorporates the average power flow associated with the reflected, transmitted, and trapped wave fields. This approach has two main advantages; first, by following the time averages power flow associated with the modes, it creates dominant diagonal terms in the final matrix equation which leads to fast convergence; second, it transforms the stress and displacement equations at the interface into the equations representing the power flow balance across the interface between separate wave fields. In short, this approach incorporates the balance of power flow across the interface which ensures fast convergence.

The projection vector space for Equation (14) is

\[
\bar{u}_{1b} = \text{conj} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{1b,\pi}, \quad \bar{u}_{1b} = 1, 2, 3, \ldots
\]

After projecting Equation (14) onto Equation (18), the stress boundary conditions in Equation (14) take the form,

\[
\int_{-h_1/2}^{h_1/2} (\bar{\sigma}_0 + \bar{\sigma}_1) \cdot \bar{u}_{1b} dy = \int_{-h_1/2}^{h_1/2} \bar{\sigma}_2 \cdot \bar{u}_{1b} dy + \int_{h_1/2}^{h_2/2} \bar{\sigma}_3 \cdot \bar{u}_{1b} dy
\]

\[
\Rightarrow \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} C_{2F,n_2} \left[ \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{2F,n_2} \right]_{1b,\pi}^{h_2-h_1/2} + C_{2B,n_2} \left[ \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{2B,n_2} \right]_{1b,\pi}^{h_1/2} + C_{3B,n_2} \left[ \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{3B,n_2} \right]_{1b,\pi}^{h_3/2}
\]

\[
\int_{a}^{b} P \cdot Q dy = \langle P, Q \rangle_{a}^{b}
\]

where, \( \int_{a}^{b} P \cdot Q dy = \langle P, Q \rangle_{a}^{b} \) represents the inner product. Similarly, the projection vector space for Equation (16) is

\[
\bar{u}_{4b} = \text{conj} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{4b,\pi}, \quad \bar{u}_{4b} = 1, 2, 3, \ldots
\]

Upon projecting Equation (16) onto Equation (20), the stress boundary conditions in Equation (16) take the form.
\[
\int_{-h/2}^{h/2} \mathbf{\bar{\sigma}}_4 \cdot \mathbf{\bar{u}}_{4\text{F}} \, dy = \int_{-h/2}^{h/2} \mathbf{\bar{\sigma}}_2 \cdot \mathbf{\bar{u}}_{4\text{F}} \, dy + \int_{-h/2}^{h/2} \mathbf{\bar{\sigma}}_3 \cdot \mathbf{\bar{u}}_{4\text{F}} \, dy
\]

\[
\Rightarrow \sum_{n_2=1}^{\infty} \left( C_{2F,n_2} \left\langle \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}, \begin{bmatrix} \mathbf{\bar{u}}_x \\ \mathbf{\bar{u}}_y \end{bmatrix} \right\rangle_{1,2F,n_2}^{h/2} + C_{2B,n_2} \left\langle \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}, \begin{bmatrix} \mathbf{\bar{u}}_x \\ \mathbf{\bar{u}}_y \end{bmatrix} \right\rangle_{1,2B,n_2}^{h/2} \right)
\]

\[
+ \sum_{n_3=1}^{\infty} \left( C_{3F,n_3} \left\langle \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}, \begin{bmatrix} \mathbf{\bar{u}}_x \\ \mathbf{\bar{u}}_y \end{bmatrix} \right\rangle_{1,3F,n_3}^{h/2} + C_{3B,n_3} \left\langle \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}, \begin{bmatrix} \mathbf{\bar{u}}_x \\ \mathbf{\bar{u}}_y \end{bmatrix} \right\rangle_{1,3B,n_3}^{h/2} \right)
\]

\[
= \sum_{n_4=1}^{\infty} \left( C_{4F,n_4} \left\langle \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}, \begin{bmatrix} \mathbf{\bar{u}}_x \\ \mathbf{\bar{u}}_y \end{bmatrix} \right\rangle_{1,4F,n_4}^{h/2} \right) ; \quad n_2, n_3, n_4 = 1, 2, 3, \ldots
\]

The projection vector space for Equation (15) is:

\[
\mathbf{\bar{\sigma}}_{3F} = \text{conj} \left[ \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix} \right]_{3F,n_3}^{F} = \left[ \begin{bmatrix} \bar{\sigma}_{xx} \\ \bar{\tau}_{xy} \end{bmatrix} \right]_{3F,n_3}^{F} , \quad \bar{n}_3 = 1, 2, 3, \ldots
\]

After projecting Equation (15) onto Equation (22), the displacement boundary conditions in Equation (15), take the form

\[
\int_{-h/2}^{h/2} \left( \mathbf{\bar{u}}_0 + \mathbf{\bar{u}}_1 \right) \cdot \mathbf{\bar{\sigma}}_{3F} \, dy = \int_{-h/2}^{h/2} \mathbf{\bar{u}}_3 \cdot \mathbf{\bar{\sigma}}_{3F} \, dy
\]

\[
\Rightarrow \sum_{n_1=1}^{\infty} \left( C_{3F,n_1} \left\langle \begin{bmatrix} \mathbf{\bar{u}}_x \\ \mathbf{\bar{u}}_y \end{bmatrix}_{3F,n_1}^{1}, \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{3F,n_1}^{1} \right\rangle_{1,3F,n_1}^{h/2} + C_{3B,n_1} \left\langle \begin{bmatrix} \mathbf{\bar{u}}_x \\ \mathbf{\bar{u}}_y \end{bmatrix}_{3B,n_1}^{1}, \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{3B,n_1}^{1} \right\rangle_{1,3B,n_1}^{h/2} \right)
\]

\[
- \sum_{n_1=1}^{\infty} C_{1B,n_1} \left\langle \begin{bmatrix} \mathbf{\bar{u}}_x \\ \mathbf{\bar{u}}_y \end{bmatrix}_{1B,n_1}^{1}, \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{1B,n_1}^{1} \right\rangle_{1,1B,n_1}^{h/2} = \left\langle \begin{bmatrix} \mathbf{\bar{u}}_x \\ \mathbf{\bar{u}}_y \end{bmatrix}_{1F,n_1}^{0}, \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{1F,n_1}^{0} \right\rangle_{1,1F,n_1}^{h/2} \quad n_1, n_3 = 1, 2, 3, \ldots
\]

The projection vector space for Equation (44) is:

\[
\mathbf{\bar{\sigma}}_{2F} = \text{conj} \left[ \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix} \right]_{2F,n_2}^{F} = \left[ \begin{bmatrix} \bar{\sigma}_{xx} \\ \bar{\tau}_{xy} \end{bmatrix} \right]_{2F,n_2}^{F} , \quad \bar{n}_2 = 1, 2, 3, \ldots
\]

After projecting Equation (44) onto Equation (22), the displacement boundary conditions in Equation (44), take the form

\[
\int_{-h/2}^{h/2} \left( \mathbf{\bar{u}}_0 + \mathbf{\bar{u}}_1 \right) \cdot \mathbf{\bar{\sigma}}_{2F} \, dy = \int_{-h/2}^{h/2} \mathbf{\bar{u}}_3 \cdot \mathbf{\bar{\sigma}}_{2F} \, dy
\]

\[
\Rightarrow \sum_{n_1=1}^{\infty} \left( C_{2F,n_1} \left\langle \begin{bmatrix} \mathbf{\bar{u}}_x \\ \mathbf{\bar{u}}_y \end{bmatrix}_{2F,n_1}^{1}, \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{2F,n_1}^{1} \right\rangle_{1,2F,n_1}^{h/2} + C_{2B,n_1} \left\langle \begin{bmatrix} \mathbf{\bar{u}}_x \\ \mathbf{\bar{u}}_y \end{bmatrix}_{2B,n_1}^{1}, \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{2B,n_1}^{1} \right\rangle_{1,2B,n_1}^{h/2} \right)
\]

\[
- \sum_{n_1=1}^{\infty} C_{1B,n_1} \left\langle \begin{bmatrix} \mathbf{\bar{u}}_x \\ \mathbf{\bar{u}}_y \end{bmatrix}_{1B,n_1}^{1}, \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{1B,n_1}^{1} \right\rangle_{1,1B,n_1}^{h/2} = \left\langle \begin{bmatrix} \mathbf{\bar{u}}_x \\ \mathbf{\bar{u}}_y \end{bmatrix}_{1F,n_1}^{0}, \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{1F,n_1}^{0} \right\rangle_{1,1F,n_1}^{h/2} \quad n_1, n_2 = 1, 2, 3, \ldots
\]
The projection vector space for Equation (17) is

\[
\overline{\sigma}_{3b} = \text{conj} \begin{bmatrix} \sigma_{xx} \\ \overline{r}_{xy} \end{bmatrix}_{3B, \overline{n}_3} = \begin{bmatrix} \sigma_{xx} \\ \overline{r}_{xy} \end{bmatrix}_{3B, \overline{n}_3}, \quad \overline{n}_3 = 1, 2, 3, \ldots
\]  

(56)

Upon projecting Equation (17) onto Equation (24), the displacement boundary conditions in Equation (17) take the form

\[
\int_{h_2/2}^{h/2} \overline{u}_4 \cdot \overline{\sigma}_{3b} \, dy = \int_{h_2/2}^{h/2} \overline{u}_3 \cdot \overline{\sigma}_{3b} \, dy
\]

\[
\Rightarrow \sum_{n_3=1}^{\infty} \left( C_{3F, n_3} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{3F, n_3}, \begin{bmatrix} \sigma_{xx} \\ \overline{r}_{xy} \end{bmatrix}_{3B, \overline{n}_3} \right)_{h_2/2}^{h/2} + C_{3B, n_3} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{3B, n_3}, \begin{bmatrix} \sigma_{xx} \\ \overline{r}_{xy} \end{bmatrix}_{3B, \overline{n}_3} \right)_{h_2/2}^{h/2}
\]

\[
= \sum_{n_3=1}^{\infty} C_{3F, n_4} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{3F, n_4}, \begin{bmatrix} \sigma_{xx} \\ \overline{r}_{xy} \end{bmatrix}_{3B, \overline{n}_3} \right)_{h_2/2}^{h/2}
\]

\[
\Rightarrow \sum_{n_3=1}^{\infty} \left( C_{3F, n_3} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{3F, n_3}, \begin{bmatrix} \sigma_{xx} \\ \overline{r}_{xy} \end{bmatrix}_{3B, \overline{n}_3} \right)_{h_2/2}^{h/2}
\]

\[
= \sum_{n_3=1}^{\infty} C_{3F, n_4} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{3F, n_4}, \begin{bmatrix} \sigma_{xx} \\ \overline{r}_{xy} \end{bmatrix}_{3B, \overline{n}_3} \right)_{h_2/2}^{h/2}
\]

(57)

The projection vector space for Equation (47) is

\[
\overline{\sigma}_{2b} = \text{conj} \begin{bmatrix} \sigma_{xx} \\ \overline{r}_{xy} \end{bmatrix}_{2B, \overline{n}_2} = \begin{bmatrix} \sigma_{xx} \\ \overline{r}_{xy} \end{bmatrix}_{2B, \overline{n}_2}, \quad \overline{n}_2 = 1, 2, 3, \ldots
\]  

(58)

Upon projecting Equation (47) onto Equation (24), the displacement boundary conditions in Equation (47) take the form

\[
\int_{-h/2}^{-h_2/2} \overline{u}_4 \cdot \overline{\sigma}_{2b} \, dy = \int_{-h/2}^{-h_2/2} \overline{u}_2 \cdot \overline{\sigma}_{2b} \, dy
\]

\[
\Rightarrow \sum_{n_4=1}^{\infty} \left( C_{2F, n_4} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{2F, n_4}, \begin{bmatrix} \sigma_{xx} \\ \overline{r}_{xy} \end{bmatrix}_{2B, \overline{n}_2} \right)_{-h/2}^{-h_2/2} + C_{2B, n_4} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{2B, n_4}, \begin{bmatrix} \sigma_{xx} \\ \overline{r}_{xy} \end{bmatrix}_{2B, \overline{n}_2} \right)_{-h/2}^{-h_2/2}
\]

\[
= \sum_{n_4=1}^{\infty} C_{2F, n_4} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{2F, n_4}, \begin{bmatrix} \sigma_{xx} \\ \overline{r}_{xy} \end{bmatrix}_{2B, \overline{n}_2} \right)_{-h/2}^{-h_2/2}
\]

\[
\Rightarrow \sum_{n_4=1}^{\infty} \left( C_{2F, n_4} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{2F, n_4}, \begin{bmatrix} \sigma_{xx} \\ \overline{r}_{xy} \end{bmatrix}_{2B, \overline{n}_2} \right)_{-h/2}^{-h_2/2}
\]

\[
= \sum_{n_4=1}^{\infty} C_{2F, n_4} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{2F, n_4}, \begin{bmatrix} \sigma_{xx} \\ \overline{r}_{xy} \end{bmatrix}_{2B, \overline{n}_2} \right)_{-h/2}^{-h_2/2}
\]

(59)

\[
n_2, n_4 = 1, 2, 3, \ldots
\]

3.4.3 Numerical Solution

For numerical calculation we consider finite values for the indices \( n_1, n_2, n_3, n_4, \overline{n}_1, \overline{n}_2, \overline{n}_3, \overline{n}_4 \). We assume,

\[
n_1 = 1, 2, 3, \ldots, N_1, \quad n_2 = 1, 2, 3, \ldots, N_2, \quad n_3 = 1, 2, 3, \ldots, N_3, \quad n_4 = 1, 2, 3, \ldots, N_4, \quad \overline{n}_1 = 1, 2, 3, \ldots, \overline{N}_1, \quad \overline{n}_2 = 1, 2, 3, \ldots, \overline{N}_2, \quad \overline{n}_3 = 1, 2, 3, \ldots, \overline{N}_3, \quad \overline{n}_4 = 1, 2, 3, \ldots, \overline{N}_4.
\]

Then, Equation (19) contains \( \overline{N}_1 \) linear equations with \( (N_1 + 2N_2 + 2N_3) \) unknowns and Equation (21) contains \( \overline{N}_1 \) linear equations with \( (N_4 + 2N_2 + 2N_3) \) unknowns. Also Equation (53) contains \( \overline{N}_3 \) linear equations with \( (N_1 + 2N_3) \) unknowns, Equation (57) contains \( \overline{N}_3 \) linear equations with \( (N_4 + 2N_3) \) unknowns, and Equation (25) contains \( \overline{N}_2 \) linear equations with \( (N_4 + 2N_2) \) unknowns. Recall that the \( (N_1 + 2N_2 + 2N_3 + N_4) \) unknowns are the complex Lamb wave mode amplitudes
Thus, Equations (19), (21), (53), (23), (57), (25) combine to form a set of \((\bar{N}_1 + 2\bar{N}_2 + 2\bar{N}_3 + \bar{N}_4)\) linear algebraic equations in \((N_1 + 2N_2 + 2N_3 + N_4)\) unknowns \(C_{1B,n_1}\), \(C_{2B,n_2}\), \(C_{3B,n_3}\), \(C_{4B,n_4}\). By assuming \(N_1 = N_2 = N_3 = N_4 = \bar{N}_1 = \bar{N}_2 = \bar{N}_3 = \bar{N}_4 = N\), we get \(6N\) equations in \(6N\) unknowns. Then Equations (19), (21) can be written as

\[
[A]_{6N \times N} \{C_{2F}\}_{N \times 1} + [B]_{6N \times N} \{C_{2B}\}_{N \times 1} + [D]_{6N \times N} \{C_{3F}\}_{N \times 1} + [E]_{6N \times N} \{C_{3B}\}_{N \times 1} - [F]_{6N \times N} \{C_{1B}\}_{N \times 1} - [0]_{6N \times N} \{C_{4F}\}_{N \times 1} = \{G\}_{N \times 1} \quad (60)
\]

Similarly Equations (23) through (25) can be written as

\[
[H]_{6N \times N} \{C_{2F}\}_{N \times 1} + [J]_{6N \times N} \{C_{2B}\}_{N \times 1} + [K]_{6N \times N} \{C_{3F}\}_{N \times 1} + [L]_{6N \times N} \{C_{3B}\}_{N \times 1} - [0]_{6N \times N} \{C_{1B}\}_{N \times 1} - [M]_{6N \times N} \{C_{4F}\}_{N \times 1} = \{0\}_{N \times 1} \quad (61)
\]

Similarly Equations (26) through (25) can be written as

\[
[R]_{6N \times N} \{C_{2F}\}_{N \times 1} + [S]_{6N \times N} \{C_{2B}\}_{N \times 1} + [0]_{6N \times N} \{C_{3F}\}_{N \times 1} + [0]_{6N \times N} \{C_{3B}\}_{N \times 1} - [P]_{6N \times N} \{C_{1B}\}_{N \times 1} - [0]_{6N \times N} \{C_{4F}\}_{N \times 1} = \{Q\}_{N \times 1} \quad (62)
\]

Similarly Equations (28) through (25) can be written as

\[
[T]_{6N \times N} \{C_{2F}\}_{N \times 1} + [0]_{6N \times N} \{C_{2B}\}_{N \times 1} + [V]_{6N \times N} \{C_{3F}\}_{N \times 1} + [W]_{6N \times N} \{C_{3B}\}_{N \times 1} - [0]_{6N \times N} \{C_{1B}\}_{N \times 1} - [X]_{6N \times N} \{C_{4F}\}_{N \times 1} = \{0\}_{N \times 1} \quad (63)
\]

Similarly Equations (30) through (25) can be written as

\[
[U]_{6N \times N} \{C_{2F}\}_{N \times 1} + [Z]_{6N \times N} \{C_{2B}\}_{N \times 1} + [0]_{6N \times N} \{C_{3F}\}_{N \times 1} + [0]_{6N \times N} \{C_{3B}\}_{N \times 1} - [0]_{6N \times N} \{C_{1B}\}_{N \times 1} - [Y]_{6N \times N} \{C_{4F}\}_{N \times 1} = \{0\}_{N \times 1} \quad (64)
\]

In Equations (26), (27), (28), (29), (64), (65) the coefficient matrices \([A]_{6N \times 6N}, [B]_{6N \times 6N}, [D]_{6N \times 6N}, [E]_{6N \times 6N}, [F]_{6N \times 6N}, [G]_{6N \times 6N}, [H]_{6N \times 6N}, [J]_{6N \times 6N}, [K]_{6N \times 6N}, [L]_{6N \times 6N}, [M]_{6N \times 6N}, [O]_{6N \times 6N}, [P]_{6N \times 6N}, [Q]_{6N \times 6N}, [R]_{6N \times 6N}, [S]_{6N \times 6N}, [T]_{6N \times 6N}, [U]_{6N \times 6N}, [V]_{6N \times 6N}, [W]_{6N \times 6N}, [X]_{6N \times 6N}, [Y]_{6N \times 6N}, [Z]_{6N \times 6N}, [\Gamma]_{6N \times 6N}\) are known matrices containing the vector-projected boundary conditions; the vectors \([C_{2F}]_{6N \times 1}, [C_{2B}]_{6N \times 1}, [C_{3F}]_{6N \times 1}, [C_{3B}]_{6N \times 1}, [C_{1B}]_{6N \times 1}, [C_{4F}]_{6N \times 1}\) contain the unknown coefficients. Combining Equations (26) through (65) yields them we get

\[
\begin{bmatrix}
A & B & D & E & -F & 0 \\
H & J & K & L & 0 & -M \\
0 & 0 & N & O & -P & 0 \\
R & S & 0 & 0 & -T & 0 \\
0 & 0 & V & W & 0 & -X \\
Y & Z & 0 & 0 & 0 & -\Gamma \\
\end{bmatrix}
\begin{bmatrix}
C_{2F} \\
C_{2B} \\
C_{3F} \\
C_{3B} \\
C_{1B} \\
C_{4F} \\
\end{bmatrix}
= 
\begin{bmatrix}
G \\
0 \\
Q \\
U \\
0 \\
0 \\
\end{bmatrix}
\Rightarrow 
[\Theta]_{6N \times 6N} [C]_{6N \times 1} = [A]_{6N \times 1} \quad (66)
\]

Equation (30) can be solved for the unknown amplitudes of the reflected and transmitted Lamb wave modes as

\[
[C]_{6N \times 1} = [\Theta]^{-1} [A]_{6N \times 1} \quad (67)
\]

4 COMPARISON WITH FEM

FEM models were built to compare with the CMEP results for various damage types. The schematic of the FEM model for the horizontal damage case is illustrated in Figure 6. The non-reflective boundaries were used to damp out the wave modes close to the boundary and eliminate boundary reflections. A similar model without any damage was created as well. Comparing the displacement fields from the FEM models with a damaged and without a damage gave us the amplitudes of the scattered S0 and A0 wave modes. This process was repeated for a vertical notch and a
vertical crack as well. We considered aluminum as the material of the plate with $E = 70$ GPa, $\rho = 2780$ kg/m$^3$, $\nu = 0.33$. The comparison between the results from CMEP and FEM are presented in Fig. 7, Fig. 8, and Figure 9 for a vertical notch, a vertical crack, and a horizontal crack. We can see that for all these damage cases FEM results follow CMEP results. This confirms that CMEP can simulate all possible damage types in plate reliably and accurately.

However, the convergence of FE models is quite expensive in terms of computational time; FEM takes about 200 times more computational time than CMEP to compute the same results. This disparity in computational time is due to the fact that FEM requires very fine discretization for convergence of both phase and amplitude. It is very important to obtain correct scatter coefficients quickly for NDE and SHM. Therefore, with fast, accurate, and reliable prediction of the scatter field, CMEP can be an important tool to solve the inverse problem of detection and characterization for thin walled structures.

5 CONCLUSION

We demonstrated that the full Lamb wave complex eigen space can be used efficiently to project the interface boundary conditions for a normal modes representation of the scattered field in the case of a horizontal crack or disbond. We demonstrated that CMEP can capture the reflected and transmitted wave amplitudes as accurately as FEM. We have also shown that both CMEP and FEM can capture the characteristic resonance of horizontal cracks. However, for FEM the accuracy of the results requires higher discretization with a much higher computational cost at higher frequencies [1]. The same calculations can be done using CMEP in significantly lower computation time and with better accuracy. At high frequencies, when many propagating Lamb wave modes exist, CMEP is a better choice than FEM because it permits the accurate and reliable computation of individual amplitudes and phases of the many reflected and transmitted modes. Also, as a byproduct, our CMEP method also yields the local vibration field near the damage which is dominated by the evanescent and complex wave modes.

6 REFERENCES


Figure 1: (a) Real, (b) imaginary, and (c) complex roots of the Rayleigh-Lamb equation for $\nu = 0.33$

Fig. 2: Schematic of Lamb waves interacting with a notch

Fig. 3: Boundary conditions at (a) the notch, (b) the horizontal crack
Figure 4: Schematic of Lamb waves interacting with a horizontal crack or disbond

Figure 5: Variation of $\lambda/L_{TIM}$ with frequency

Figure 6: Schematics of the finite element model with nonreflecting boundary for harmonic analysis
Fig. 7: Comparison of \( u_x \) displacement of (a) reflected modes, (b) transmitted modes for incident S0 mode scattered from a vertical notch with \( R_d = 0.5 \) and \( R_w = 0.5 \)

Fig. 8: Comparison of \( u_x \) displacement of (a) reflected modes, (b) transmitted modes for incident S0 mode scattered from a vertical crack with \( R_d = 0.5 \)
Figure 9: Comparison of $u_x$ displacement of (a) transmitted S0 mode amplitude, (b) transmitted A0 mode amplitude, (c) reflected S0 mode amplitude, (d) reflected A0 mode amplitude for incident S0 mode scattered from a disbond with $R_d = 0.25$ and $R_w = 5$.