Towards an Unified Approach for Guided Ultrasonic Wave Dispersion Curves in Metallic and Composites Materials

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ABSTRACT

In current literature, algorithms for predicting guided ultrasonic wave dispersion curves use different solution approaches based on the material type of the medium in which the waves propagate, thereby distinguishing between isotropic and anisotropic material. For composites which have a slight degree of anisotropy, however, both solution approaches are not satisfactory. This manuscript, therefore, proposes an unified approach which is valid for any material regardless its degree of anisotropy.

The proposed unified approach was based on the eigenvalue problem which was derived from the Christoffel equation for a lamina. The eigenvalues, and eigenvectors were obtained straightforward by solving the eigenvalue problem. Special attention, however, was required when identify the polarization of the eigenvectors, and ensuring continuity within a set of eigenvectors when transiting from complex to real roots as result of the Christoffel equation. The traction free boundary conditions were applied to set up the six by six stress-displacement matrix which was solved to find the wavenumber-velocity pairs which yielded a solution. As result of using the Christoffel equation, a complex determinant for each wavenumber-velocity pair was obtained. A simple, and elegant approach, based on the phase between succeeding velocities for a fixed wavenumber, was developed, and applied successfully to determine the wavenumber-velocity pairs which yielded a solution.

The proposed unified approach was evaluated for both isotropic, and anisotropic materials. The developed algorithm verified using the commercially available software package DISPERSE.

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INTRODUCTION

There is an increasing desire from the aerospace industry to reduce maintenance while at the same time not impairing structural integrity, and safety of an aircraft. To realize this objective more information, preferably in real time, regarding the condition of a structure is required. The information need to be acquired in a non-destructive method, as fast as possible, using a minimum number of sensors, with high accuracy in a reproducible fashion. This has led to a sub-branch of non-destructive evaluation which is known as structural health monitoring (SHM) [1]. Within the field of SHM guided ultrasonic waves (GUW), especially those described by Lamb [2], are preferred.

Lamb waves are ultrasonic acoustic waves which propagate in a thin plate like, or small curvature shell like structures with two parallel free boundaries [2]. Lamb waves are the preferred wave type in SHM due to their small amplitude attenuation over long distances [3–5]. The dispersive nature of Lamb waves together with the existing of multiple modes simultaneously, however, can complicate the analysis, and therefore need to be addressed.

For a certain excitation frequency symmetric and antisymmetric Lamb waves, and shear horizontal waves exist simultaneously, abbreviated by S-waves, A-waves, SH-waves respectively. The S-waves consists of longitudinal compression-traction, in-plane motion, waves; A-waves are transverse, out-of-plane motion, waves; and the SH-waves are shear motion in the horizontal plane waves. In addition, the wave modes differ from one another by their propagation velocity, both group and phase velocities, and by their modeshape in the material thickness direction, both displacement and stress modeshapes. For isotropic materials, metals, the analysis is relatively simple; the different waves have the same propagation velocity in each direction, and the SH-waves are decoupled from the Lamb waves. In general Lamb wave propagation in metals is extensively described, and understood in the literature, such as by Viktorov [6], Rose [7], and Giurgiutiu [8] just to name a few.

The same, however, is not valid not anisotropic materials, like composites, even though composites are increasingly selected over metals as the material of choice by aircraft manufacturers. For composite materials the propagation velocities are depended on the propagation direction, and in the general case no decoupling between the SH-waves, and the Lamb waves is possible. There are, however, different methods which can be used to obtain the dispersive curves in composites, e.g. transfer matrix method (TMM) [9], global matrix method (GMM) [10], stiffness matrix method (SMM) [11], and stiffness transfer matrix method (STMM) [12]. In specific cases, however, the Lamb waves can be decoupled from the SH-waves as shown by Nayfeh [13]. Nayfeh [13] formulated an analytic solution for n-layered monoclinic lamina which was solved using the TMM [9]. Lowe et al. [14], on the other hand, used the GMM to solve the problem for n-layered orthotropic composites, this led to the commercially available package called DISPERSE.

Depending on the material type a different solution procedure is required, this can be inconvenient when both type of material are used in the structure, e.g. fiber metal laminates. Or when dealing with a material which has a slight degree of anisotropy, where both solution procedure, theoretically, can be applied. In this manuscript an unified solution procedure, based on Christoffels equation for a lamina, therefore, is presented.
which can be applied to any material regardless its degree of anisotropy, or to composites which include both isotropic and anisotropic material. This manuscript is structured as such: first, the theoretical background used is elaborated upon; second, the numerical implementation and the results are discussed; and as last, the conclusions are presented.

**THEORETICAL BACKGROUND**

This section provides a concise overview of the theoretical background, for a comprehensive understanding of the applied theory the reader is advised to read [15]. The proposed unified approach was based on the eigenvalue problem, given in Eq. 1, derived from the Christoffel equation for a lamina.

\[
\begin{bmatrix}
(C_{11} - \rho v^2) + C_{55} \alpha^2 & C_{16} + C_{45} \alpha^2 & (C_{13} + C_{55}) \alpha^2 \\
C_{16} + C_{45} \alpha^2 & (C_{66} - \rho v^2) + C_{44} \alpha^2 & (C_{36} + C_{45}) \alpha^2 \\
(C_{13} + C_{55}) \alpha^2 & (C_{36} + C_{45}) \alpha^2 & (C_{55} - \rho v^2) + C_{33} \alpha^2
\end{bmatrix}
\begin{bmatrix}
\hat{u}_1 \\
\hat{u}_2 \\
\hat{u}_3
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\tag{1}
\]

The determinant of Eq. 1 was expanded, and reformulated as a cubic formula before solving for the eigenvalue pairs first, and corresponding eigenvectors second. Using the eigenvalues and eigenvectors the field matrix for the displacement, defined by Eq. 2, was calculated. Where \( b^j_u(x_3) \) is given by Eq. 3.

\[
B^u(x_3) = [b^1_u(x_3) \ b^2_u(x_3) \ b^3_u(x_3) \ b^4_u(x_3) \ b^5_u(x_3)] \tag{2}
\]

\[
b^j_u(x_3) = U^j_3 e^{+i\xi \alpha^j x_3} \tag{3}
\]

For the stress field matrix the monoclinic lamina under \( x_2 \)-invariant conditions were used, which resulted in Eq. 4. Where \( b^j_\sigma(x_3) \) is given by Eq. 5.

\[
B^\sigma(x_3) = [b^1_\sigma(x_3) \ b^2_\sigma(x_3) \ b^3_\sigma(x_3) \ b^4_\sigma(x_3) \ b^5_\sigma(x_3)] \tag{4}
\]

\[
b^j_\sigma(x_3) = i \xi \begin{bmatrix}
C_{13} U^j_1 + C_{33} \alpha^j U^j_3 + C_{36} U^j_2 \\
C_{44} \alpha^j U^j_2 + C_{45} U^j_3 + C_{45} \alpha^j U^j_3 \\
C_{45} \alpha^j U^j_2 + C_{55} U^j_3 + C_{55} \alpha^j U^j_3
\end{bmatrix} e^{+i\xi \alpha^j x_3} \tag{5}
\]

To obtain the dispersion curves the traction-free boundary conditions were applied to Eq. 4 to obtain the field matrix \( B^\sigma \) given in Eq. 6. All the possible wavenumber-velocity combinations which resulted in a sign change in the determinant of Eq. 6 yielded desired dispersion curves of the material.

\[
\begin{bmatrix}
B^\sigma(0) \\
B^\sigma(h)
\end{bmatrix} \eta = 0 \tag{6}
\]

Note, \( 0 \) and \( h \) represent the \( x_3 \)-coordinates of the bottom and top surface which are traction free. After the eigenvectors, \( \eta \), of Eq. 6 were determined the modeshapes were calculated using Eq. 7.
NUMERICAL IMPLEMENTATION AND RESULTS

The algorithm was tested using an isotropic material, and anisotropic material, the corresponding stiffness matrices are given in GPa by Eq. 8. For isotropic material of the eigenvalues belonging to the shear horizontal and shear vertical waves were the same, this behavior was not recorded for an anisotropic material, as shown in Figure 1. In addition, it can be seen that the squared of the eigenvalues depending on the velocity are negative, or positive. This implies that eigenvalues change from the complex to the real domain at a certain velocity.

\[
\begin{bmatrix}
\hat{u}_k(x_3) \\
\hat{\sigma}_k(x_3)
\end{bmatrix}
= 
\begin{bmatrix}
B^u_k(x_3)\eta_k \\
B^\sigma_k(x_3)\eta_k
\end{bmatrix}
\]

(Eq. 7)

Depending on the domain of the eigenvalues the corresponding eigenvectors were calculated using different MATLAB subroutines. For complex domain \textit{eig}-function was used, while for the real domain singular value decomposition, the \textit{svd}-function, was used to approximate the actual eigenvector. Two different subroutines were needed to ensure magnitude continuity within the eigenvector at different velocities. In addition to the magnitude continuity, a simple algorithm was developed to check for sign continuity, and enforced it, by multiplying by minus or plus one respectively, when required. Both type continuities within the eigenvectors were of great importance when searching for wavenumber-velocity pairs which led to a sign change of the determinant of the field matrix \(B\), and therefore, a solution to the dispersion curves. Without continuity the sign of the determinant changed for incorrect wavenumber-velocity pairs, which lead to false
positives or negatives when the dispersion curves were retrieved. Due to the formulation of the problem the determinant of the field matrix $\mathbf{B}$ was in the complex domain. Therefore, to retrieve the dispersion curves both the real part, and the imaginary part of the determinant had to change sign simultaneously for a wavenumber-velocity pair to yield a solution. Multiple approaches can be taken to search for the wavenumber-velocity pairs, in this investigation a simple and elegant approach was used. The phase angle between determinants was used instead of searching for sign change of the real and imaginary part separately, and checking when the sign change occurred at the same wavenumber-velocity pair. The phase angle approach was able to retrieve the wavenumber-velocity pairs by evaluating if the change in phase angle of two consecutive determinants was within a certain range, Eq. 9 summarizes the phase angle approach mathematically.

\[
\Lambda = \theta_i - \theta_{i+1} \\
(180 - \text{mod}(\theta_i, 90)) \leq \Lambda \leq (270 - \text{mod}(\theta_i, 90))
\]  

For isotropic materials, as shown in Figure 2a, areas are present were the no sign change was retrieved while one was expected. Investigating these areas revealed that multiple wave modes were overlapping each other which doubled the phase angle, and thereby violating the statement given in Eq. 9. Nevertheless, the straight forward, and simple formulation of the problem yielded the dispersion curves for the Lamb waves, and the SH-waves simultaneously using one unified approach regardless the material type. The dispersion curves were verified using DISPERSE, the comparison is given in Figure 3. It is important to note that only the first five waves were compared, and the SHS0-wave was not obtained when using DISPERSE.

As last, Eq. 7 was used to obtain the modeshapes of the different waves types. The modeshapes of the waves were used to determine the type of the wave. To be concise, this manuscript only focuses on the fundamental Lamb waves, S0 and A0, and the fundamental SH-waves, SHS0 and SHA0. In Figure 4 and Figure 5 the displacement modeshapes for the fundamental waves at a wavenumber of 1810 are given for an isotropic and anisotropic material, respectively. Figure 4 revealed an unexpected behavior for the SHS0 waves. The expected behavior for an SHS0 was amplitude in the $U_2$
direction while the amplitude in $U_1$ and $U_3$ were zero, it is important to notice that $U_1$, $U_2$, $U_3$ correspond to the $x_1$, $x_2$, $x_3$ direction of the material. A closer examination of this case revealed that the problem originated at the corresponding eigenvectors. Each eigenvector is a six by one vector, due to the problem setup an SH-eigenvector should have non-zeroes for the first two rows while the others rows are zeroes. In this case, however, the SHS0-eigenvector had a same form as a Lamb wave eigenvector; zeroes for the first two rows while the others rows were non-zeroes. Nevertheless, the dot product between the modeshapes was used to identify the waves modes, the corresponding result is shown in Figure 6, the SHS0 and higher order modes were neglected for simplicity. After the waves were identified a spline was used to obtained a continuous solution over the entire wavenumber domain.

CONCLUSION

In this manuscript an unified formulation for guided ultrasonic wave dispersion curves in isotropic and anisotropic materials was introduced. The numerical implementation of the formulation was discussed, and initial results were presented. The obtained dispersion curves were compared with that of DISPERSE showing that the unified formulation was capable to produce the correct results.

The analysis of the displacement modeshapes revealed some modeshapes were different than expected. The difference between the expected and obtained modeshapes originated from the corresponding eigenvectors that had a different form than expected. From the unified formulation it was expected that SH-waves had an eigenvector of which the first two rows were non-zeroes while the other four rows were zeroes. This type of eigenvector was not obtained for an isotropic material, for anisotropic material, however, the desired eigenvector and corresponding modeshape were obtained. Finally, the dot product between the modeshapes was used to identify the waves modes.

The eigenvectors, the building block of the unified formulation, need extra attention. Future work will focus on obtaining eigenvectors with increased numerical accuracy and ensuring continuity between the eigenvectors of various modes at various wavenumbers.
Figure 4: A0, S0, SHA0, SHS0 modeshapes for an isotropic material at wavenumber $\xi=1810/m$

Figure 5: A0, S0, SHA0, SHS0 modeshapes for an anisotropic material at wavenumber $\xi=1810/m$
Figure 6: Dispersion curves after modeshape curve identification

This should be attained for both isotropic and anisotropic material types.

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