Simplified 2-D Modeling of Power and Energy Transduction of Piezoelectric Wafer Active Sensors for Structural Health Monitoring

Bin Lin, Victor Giurgiutiu
Mechanical Engineering Department, University of South Carolina
Columbia, SC 29208, linbin@cec.sc.edu, victorg@sc.edu

ABSTRACT

This paper presents an investigation of 2-D power and energy transduction in piezoelectric wafer active sensors (PWAS) for structural health monitoring (SHM). After a literature review of the state of the art, we developed a model of 2-D power and energy transduction of PWAS attached to structure. The model is an extension of our previously presented 1-D model. It allows examination of power and energy flow for a circular crested wave pattern. The model assumptions include: (a) 2-D axial and flexural wave propagation; (b) ideal bonding (line-force) connection between PWAS and structure; (c) ideal excitation source at the transmitter PWAS and fully-resistive external load at the receiver PWAS (d) crested wave energy spread out. Wave propagation method for an infinite boundary plate, electromechanical energy transformation of PWAS and structure, and wave propagation energy spread out in 2-D plate were considered. The parametric study of PWAS size, impedance match gives the PWAS design guideline for PWAS sensing and power harvesting applications. In pitch-catch PWAS application, the frequency response functions of a circular PWAS are developed for voltage in consideration with the receiver capacitance and external resistive loads.

Keywords: 2-D, power, energy, piezoelectric wafer active sensors

1. INTRODUCTION

1.1 Background

The mounting costs of maintaining our aging infrastructure and the associated safety issues are a growing national concern. Over 27% of our nation’s bridges are structurally deficient or functionally obsolete. Deadly accidents are still marring our everyday life. In response to these growing concerns, structural health monitoring (SHM) sets forth to determine the health of a structure by monitoring over time a set of structural sensors and assessing the remaining useful life and the need for structural actions. Built-in SHM system capable of detecting and quantifying damage would increase the operational safety and reliability, would conceivably reduce the number of unscheduled repairs, and would bring down maintenance cost.

The type and efficiency of the SHM sensors play a crucial role in the SHM system success. Ideally, SHM sensors should be able to actively interrogate the structure and find out its state of health, its remaining life, and the effective margin of safety. Essential in this determination is to find out the presence and extend of structural damage. Recent SHM work has shown that piezoelectric wafers adhesively bonded to the structure successfully emulate the NDE methodology (pitch-catch, pulse-echo, phased array, Figure 1) while being sufficiently small and inexpensive to allow permanent attachment to the monitored structure. Piezoelectric wafer active sensors (PWAS) are small, lightweight, unobtrusive, and inexpensive and achieve direct transduction between electric and elastic wave energies. Two SHM sensing principles can be considered: (a) passive SHM sensing, in which the damage of the structure is inferred from the changes in load and strain distributions measured by the sensors; and (b) active SHM sensing, in which the damage is
sensed by active interrogation of the structure with elastic waves. The power and energy flow in active and passive sensing is an important factor and has not been systematically addressed.

Figure 1 (a) pitch-catch method; (b) pulse-echo method; (c) bonded interface between PWAS and structure

1.2 State of the Art
Up to date, work on PWAS SHM technology has not yet systematically addressed the modeling of power and energy transduction. However, this topic has been addressed to a certain extent in classical NDE. Viktorov mentioned the transmissibility function between an NDE transducer and the guided waves in the structure, but did not use an analytical expression of this transmissibility function but rather relies on experimentally determined results. Auld treated comprehensively the power and energy of ultrasonic acoustic fields and developed the complex reciprocity approach to their calculation. He developed some predictive models for surface acoustic wave (SAW) devices that relied on the simplifying assumption of single mode nondispersive Rayleigh wave propagation. Rose developed a model for coupling between an angle-beam ultrasonic transducer and a system of guided Lamb waves in the structure using the normal modes expansion approach. This model was not specific about how the shear transfer takes place through the gel coupling between the transducer and the structure. The power flow between transducer and structure has not been well studied in classic NDE because (a) the coupling-gel interface did not have clearly predictable behavior; (b) power was not generally an issue, since NDE devices are not meant to operate autonomously on harvested power. Recent research for the design of autonomous SHM systems employing structurally integrated active sensors has a need of understanding and mastering of the power and energy flow. Giurgiutiu et al. established a wave tuning model of axial, flexural waves excited by PWAS and then extended this analysis to the case of a circular PWAS. Yeum and Sohn further decomposed the Lamb wave mode using concentric ring and circular piezoelectric transducer. Lin and Giurgiutiu considered ideal bonding and performed a systematic investigation of 1-D power and energy transduction in PWAS attached to an infinite beam using the simple axial and flexural wave.

1.3 Motivation
The purpose of the paper is to extend the 1-D power and energy analysis to a more realistic 2-D power model in PWAS active and passive SHM sensing. It is important for the design of autonomous SHM systems employing PWAS with optimum power and energy flow. The following issues were addressed: (a) predictive modeling of 2-D axial and flexural wave propagation; (b) the power and energy transduction under circular PWAS excitation; (c) predictive modeling of voltage frequency response function of a circular PWAS receiver under a circular PWAS excitation; (d) identification of maximum energy flow; (e) knowing that energy flows under impedance match conditions.

2. 2D CIRCULAR WAVE PROPAGATION MODEL
Figure 2 presents an isotropic plate with a PWAS transmitter and PWAS receiver bonded on a surface of the plate. Assuming the circular PWAS transmitter is with radius \( a \) located at the origin of a polar coordinate system and it produces axisymmetric circular waves under electrical excitation. A circular PWAS receiver with the center at \((r_c, 0)\) and radius \( c \) converts the waves back to electrical power. The analytical model of wave propagation and power and energy transduction is based on the following assumptions: (a) ideal bonding (line-force) connection between PWAS and structure; (b) ideal excitation source at the transmitter PWAS and fully-resistive external load at the receiver PWAS; and (c) 2-D axial and flexural wave propagation.
For compactness, the notations were used in this article:

\[ \dot{U} = \frac{\partial}{\partial t} U \quad U' = \frac{\partial}{\partial x} U \quad U = \dot{U} e^{i\omega t} \quad \overline{U} = \text{conj}(U) \]  

(2.1)

### 2.1 Line-force Model

The power excited by a circular PWAS is examined based on the 2-D linear elasticity equation. The cylindrical coordinate system is the natural choice for the analysis. The electrical field is produced by the application of a voltage between the top and bottom surface electrodes. The resulting electrical field is assumed uniform over the electrode areas. Because the electric field is uniform, the response will be assumed to be axially symmetric. This means the circular PWAS undergoes uniform radial and circumferential expansion. The linear constitutive equations for piezoelectric materials can be expressed in cylindrical coordinates in the following form (Onoe et al., 1967; Pugachev, 1984; IEEE, 1987):

\[
S_{rr} = s^{E}_{11} T_{rr} + s^{E}_{12} T_{00} + d_{31} E_{3}, \\
D_{3} = d_{33} T_{rr} + d_{33} T_{00} + e_{33} E_{3},
\]  

(2.2)

where \( S_{rr}, S_{00} \) are the strain, \( T_{rr}, T_{00} \) are the stress, \( D_{3} \) is the electrical displacement, \( s^{E}_{11}, s^{E}_{12} \) are the mechanical compliance at zero field, \( e_{33} \) is the dielectric constant at zero stress, \( d_{31} \) is the induced strain coefficient.

The actuation and sensing between the PWAS and the structure is achieved through the adhesive layer. Crawley and De Luis\(^{12}\) developed an analytical model of the coupling between piezoelectric wafer actuators and thin wall structural members. The configuration studied was of two piezoelectric elements bonded on both sides of an elastic structure. They assumed that the strain distribution in the piezoelectric actuator was a linear distribution across the thickness (Euler–Bernoulli linear flexural or uniform extension) and developed a shear lag solution for the interfacial stress between the PWAS and the structure. Giurgiuțiu\(^{6}\) extended the shear lag solution to the case of a single PWAS attached to one side of structure. Santoni-Bottai and Giurgiuțiu\(^{15,16}\) further extended this work by removing the Euler-Bernoulli assumption and considering multiple Lamb waves mode simultaneously present in the structure. They have indicated that the simplified pin-force model can be used under non-ideal conditions by employing an effective PWAS length.
which is shorter than the actual length by a percentage. In this work, we are implicitly assuming we are using effective PWAS size and considering the ideal bonding of adhesive layer. The concept of ideal bonding (also known as the pin-force model) assumed that all the load transfer takes place over an infinitesimal region at the PWAS ends, and the induced-strain action is assumed to consist of a pair of concentrated forces applied at the ends. The adhesive layer act as a shear layer, in which the mechanical effects are transmitted through shear effects. Under static and low-frequency dynamic conditions, the usual hypothesis associates with simple axial and flexural motions, i.e., constant displacement for axial motion and linear displacement strain for flexural motion. Future work will consider the power and energy with full shear-lag analysis.

The line-force model is convenient for getting simple solutions that represent a first-order of approximation to the PWAS-structure interaction in 2-D. The line-force applied by the circular PWAS circumference contributes the axial force and bending moment to the structure (Figure 3b). Considering a circular transmitter PWAS with the center at origin, radius \( a \), and thickness \( h_{PWAS} \), the line force \( F_{PWAS} \) generated by PWAS is applied on the circular PWAS circumference. The axial force and bending moment applied to the plate can be written as

\[
N^*_r = F_{PWAS} \left[ H(a - r) \right] \\
M^*_r = M \left[ H(a - r) \right] = \frac{h_{PWAS}}{2} F_{PWAS} \left[ 1 - H(r - a) \right]
\]

where \( N^*_r \) and \( M^*_r \) are the axial force and bending moment on the structure; \( H(x) \) is the Heaviside step function and \( h_{plate} \) is the thickness of the plate.

Figure 3 PWAS model (a) PWAS attached on a plate; (b) line-force model

### 2.2 Circular Crested Wave Propagation under Line-force

The axial wave field excited by a circular PWAS line-force assumes that the force generated by PWAS is only applied on the circular PWAS circumference. Due to the axisymmetry and no shear waves along the \( \theta \) direction, the equation of motion under perimeter force in cylindrical coordinates can be written as:

\[
\frac{E}{(1 - \nu_{plate})^2} \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right) - \rho_{plate} \frac{\partial^2 u}{\partial t^2} = -\frac{F_{PWAS}}{2\pi ah_{plate}} \delta(r - a)
\]

where \( \nu_{plate} \) is Young’s modulus of the plate.

Using the space-domain Hankel transform and the residual Theorem, the axial displacement can be written as

\[
u(r,t) = i\pi a \frac{E}{\xi_0} J_1(\xi_0 a) J_1(\xi_0 r) e^{\xi_0 t}
\]

where the line strain \( \xi_0 \), the wave number \( \xi_0 \), and the axial wave speed \( c_L \) are defined as

\[
\xi_0 = \frac{1 - \nu_{plate}^2}{2\pi ah_{plate} E} F_{PWAS} \quad \xi_0^2 = \frac{E}{\rho_{plate}} \quad c_L = \sqrt{\frac{E}{\rho_{plate}(1 - \nu_{plate}^2)}}
\]

Due to the axisymmetry and no shear waves along the \( \theta \) direction, the equation of flexural motion in cylindrical...
coordinates can be written as:
\[ D \nabla^4 w + \rho_{\text{plate}} h_{\text{plate}} \frac{\partial^2 w}{\partial t^2} = \frac{h_{\text{plate}}}{2} \frac{F_{\text{PWAS}}}{2\pi a} \delta'(r-a) \]
where \( \nabla^4 w \) and \( D \) is defined as
\[ \nabla^4 w = \frac{\partial^4 w}{\partial r^4} + \frac{2 \partial^4 w}{r \partial r^3} - \frac{1}{r^2} \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^3} \frac{\partial w}{\partial r} \]

\[ D = \frac{E h_{\text{plate}}}{12(1-\nu^2)} \]
Using the space-domain Hankel transform and the residual Theorem, the flexural displacement can be written as
\[ w(r,t) = i \frac{1}{8 \xi_f^2} \frac{F_{\text{PWAS}} h_{\text{plate}}}{D} J_1(\xi_f a) J_0(\xi_f r) e^{-i\omega t} \]
where the flexural wave number \( \xi_f \) and the flexural wave speed \( c_f \) are defined as
\[ \xi_f^4 = \omega^2 / c_f^2 \]
\[ c_f = \sqrt{\frac{\omega h_{\text{plate}}^2}{\rho_{\text{plate}} (1-\nu^2)}} \]

Kinematic analysis gives the horizontal displacement of a generic point \( P \) on the infinite plate surface in terms of the axial and flexural displacement as
\[ u_p(r,t) = u(r,t) - \frac{h_{\text{plate}}}{2} w'(r,t) = i \pi a \epsilon_0 \left\{ \frac{J_1(\xi_f a)}{\xi_f} J_1(\xi_f r) + \frac{J_1(\xi_f a)}{\xi_f} J_1(\xi_f r) \right\} e^{-i\omega t} \]
The mechanical power provides the power and energy for the generated axial and flexural waves. The analytical derivation shows that the time-averaged energy and power for the axial wave are
\[ \langle e_A \rangle = 2 \pi a h_\omega o^2 \langle u \bar{w} \rangle \]
\[ \langle P_{\text{axial}} \rangle = c_f \langle e_f \rangle \]
Under the Euler-Bernoulli bending assumptions, the shear deformation and rotary inertia are ignored. The time-averaged flexural wave energy and power are
\[ \langle e_f \rangle = 2 \pi a h_\omega o^2 \langle w \bar{w} \rangle \]
\[ \langle P_{\text{flexural}} \rangle = c_f \langle e_f \rangle \]
The Bessel function can be approximated as follows using the fact that it exhibits an asymptotic behavior after four or five cycles of the wavelength of the mode considered.
\[ J_0(\xi_f r) \approx \frac{2}{\pi \xi_f} \cos \left( \frac{\xi_f r - \pi}{4} \right) \]
\[ J_1(\xi_f r) \approx \frac{2}{\pi \xi_f} \cos \left( \frac{\xi_f r - 3\pi}{4} \right) \]
The total axial wave is independent with the wave propagation distance \( r \). The displacement exhibits an asymptotic behavior with \( 1/r \). The flexural wave has the same properties.

### 3. 2D CIRCULAR PWAS TRANSMITTER MODEL

Figure 4 shows the power and energy transduction schematic in the case of a transmitter PWAS. The electrical energy of the input voltage applied at the PWAS terminals is converted through piezoelectric transduction into mechanical energy that activates the expansion-contraction motion of the PWAS transducer. This motion is transmitted to the underlying structure through the shear stress in the adhesive layer at the PWAS-structure interface. As a result, ultrasonic guided waves are excited into the underlying structure. The mechanical power at the interface becomes the acoustic wave power and the generated axial and flexural waves propagate in the structure. Questions that need to be answered through predictive modeling are:
(a) How much of the applied electrical energy is converted in the wave energy?
(b) How much energy is lost through the shear transfer at the PWAS-structure interface?
(c) How much of the applied electrical energy gets rejected back into the electrical source?
(d) What are the optimal combinations of PWAS geometry, excitation frequency, and wave mode for transmitting the maximum energy as ultrasonic waves into the structure?
To perform the analysis, we developed 2-D closed form analytical expressions for the active and reactive electrical power, mechanical power in the PWAS, and ultrasonic acoustic power of the waves traveling in the structure.

### 3.1 Circular PWAS Mechanical Power

Mechanical power is equal to the force times the velocity, that is

\[
P = \frac{1}{2} F V
\]

(3.1)

When the circular PWAS transmitter is excited by an oscillatory voltage, its volume expands in phase with the voltage in accordance with the piezoelectric effect. Expansion of the PWAS mounted on the surface of the plate induces a surface reaction form the plate in the form of the line force distributed around the PWAS circumference. The reaction force, \( F_{\text{PWAS}} \), depends on the PWAS radial displacement, \( u_{\text{PWAS}}(t) \), and on the frequency-dependent dynamic stiffness, \( k_{\text{str}}(\omega) \), presented by the structure to the PWAS. The PWAS circumference radial displacement is constrained by the plate and is equal to the plate displacement at \( r = a \).

\[
F_{\text{PWAS}}(t) = k_{\text{str}}(\omega) u_{\text{plane}}(a,t)
\]

(3.2)

Under harmonic excitation, the dynamic stiffness, \( k_{\text{str}}(\omega) \) is

\[
k_{\text{str}}(\omega) = \frac{F_{\text{PWAS}}}{\dot{u}_{r}(a)} = -i \frac{2E h_{\text{plate}}}{1-v_{\text{plate}}^2} \left[ J_1^2 \left( \frac{\xi_{\text{r}} a}{\xi_0} \right) + 3 J_1 \left( \frac{\xi_{\text{r}} a}{\xi_0} \right) \right]^{-1}
\]

(3.3)

The static stiffness of a free circular PWAS, \( k_{\text{PWAS}} \), is

\[
k_{\text{PWAS}} = \frac{h_{\text{PWAS}}}{a s_{11}^E (1-v_{\text{PWAS}})}
\]

(3.4)

The dynamic stiffness ratio is defined as the ratio between \( k_{\text{str}}(\omega) \) and \( k_{\text{PWAS}} \), i.e.,

\[
\chi(\omega) = \frac{k_{\text{str}}(\omega)}{k_{\text{PWAS}}} = \frac{2E h_{\text{plate}} a s_{11}^E (1-v_{\text{PWAS}})}{i h_{\text{PWAS}} (1-v_{\text{plate}}^2)} \left[ J_1^2 \left( \frac{\xi_{\text{r}} a}{\xi_0} \right) + 3 J_1 \left( \frac{\xi_{\text{r}} a}{\xi_0} \right) \right]^{-1}
\]

(3.5)

The relation between line force and the static stiffness of a free PWAS is

\[
F_{\text{PWAS}} = k_{\text{PWAS}} (u(a) - u_{\text{ISA}}) = \frac{\chi(\omega)}{1-\chi(\omega)} k_{\text{PWAS}} u_{\text{ISA}}
\]

(3.6)

where \( u_{\text{ISA}} \) is the induced strain, i.e., \( u_{\text{ISA}} = ad_{11}^E \frac{V}{h_{\text{PWAS}}} \).

Under harmonic excitation, frequency response function (FRF) is the measure of any system’s output spectrum in response to an input signal. In PWAS embedded SHM setup, the input is the excitation voltage and the output could be voltage, current, complex power etc. The frequency response function of line force at the transmitter PWAS is

\[
\text{FRF}_{FV}(\omega) = \frac{F_{\text{PWAS}}}{V} = \frac{\chi(\omega)}{1-\chi(\omega)} k_{\text{PWAS}} ad_{11}^E \frac{V}{h_{\text{PWAS}}}
\]

(3.7)

The velocity at PWAS circumference can be derived from the displacement, i.e.,
$$v_p(r = a, t) = \xi u = -i \omega u = \omega \left(1 - v_{\text{plate}}^2 \right) F_{\text{PWAS}} \left\{ J_1^2 \left( \xi_p a \right) + 3 J_1^2 \left( \xi_F a \right) \right\} e^{-i\omega t}$$  (3.8)

Under harmonic excitation, the time-averaged power is the average amount of energy converted per unit time with continuous harmonic excitation. The time-averaged product of two harmonic variables is one half the product of one variable times the conjugate of the other. The time averaged mechanical total power under harmonic excitation for PWAS transmitter is

$$\langle P_{\text{mech}} \rangle = \pi a \bar{F}_{\text{PWAS}} \langle \omega \rangle \hat{v}(\omega, r = a)$$  (3.9)

### 3.2 Circular PWAS Electrical Power

The circular PWAS transmitter admittance was derived by Giurgiuțiu, etc., that has a form of

$$Y(\omega) = i \omega C_0 \left(1 - k_p^2 \right) \left[1 + \frac{k_p^2}{(1 - k_p^2) u_{\text{Plate}}} \right] = i \omega C_0 \left(1 - k_p^2 \right) \left[1 + \frac{1}{(1 - k_p^2) 1 - X(\omega)} \right]$$  (3.10)

Where the planar coupling coefficient, $k_p$, and circular PWAS capacitance, $C_0$, are defined as $k_p^2 = \frac{2}{1 - v_{\text{PWAS}}} k_{ij}^2$ and $C_0 = \frac{e_{33} \pi a^2}{h_{\text{PWAS}}}$

The electrical power rating, time-averaged active power and reactive power are

$$P_{\text{rating}} = \frac{1}{2} Y \hat{V}^2 = \sqrt{P_{\text{active}}^2 + P_{\text{reactive}}^2} \quad P_{\text{active}} = \frac{1}{2} Y \hat{V}^2 \quad P_{\text{reactive}} = \frac{1}{2} Y \hat{I}^2$$  (3.11)

The reactive power is the imaginary part of complex power that is not consumed and is recirculated to the power supply. The power rating is the power requirement of the power supply without distortion. In induced-strain transmitter applications, the reactive power is the dominant factor, since the transmitter impedance is dominated by its capacitive behavior. Managing high reactive power requirements is one of the challenges of using induced-strain transmitters. The active power is the power that converts to the mechanical power at the interface. The simulation shows that the time-averaged mechanical power and electrical power are equal.

$$\langle P_{\text{active}} \rangle = \langle P_{\text{mech}} \rangle$$  (3.12)

### 3.3 Circular PWAS Transmitter Parameter Analysis

It was found (Figure 5) that the reactive electrical power required for PWAS excitation is orders of magnitude larger than the active electrical power (compare Figure 5a with Figure 5b). Hence, the power rating of the PWAS transmitter is dominated by the reactive power, i.e., by the capacitive behavior of the PWAS. We note that the transmitter reactive power is directly proportional to the transmitter admittance ($Y = i \omega C$), whereas the transmitter active power is the power converted into the ultrasonic acoustic waves generated into the structure from the transmitter. A remarkable variation of active power with frequency is shown in Figure 5b: we notice that the active power (i.e., the power converted into the ultrasonic waves) is not monotonic with frequency, but manifests peaks and valleys. The increase and decrease of active power with frequency corresponds to the PWAS tuning in and out of various ultrasonic waves traveling into the structure. The maximum active power seems to be $\sim 5$ mW for a 7-mm diameter circular PWAS.

<table>
<thead>
<tr>
<th>Simulation Parameters</th>
<th>Plate structure (2024 Al alloy)</th>
<th>Transmitter PWAS (PZT-850)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius</td>
<td>$\infty$</td>
<td>2-15 mm</td>
</tr>
<tr>
<td>Height</td>
<td>1 mm</td>
<td>0.2 mm</td>
</tr>
<tr>
<td>Frequency</td>
<td>Frequency sweep 1 - 1000 kHz</td>
<td></td>
</tr>
<tr>
<td>Constant Voltage Input</td>
<td>10 V</td>
<td></td>
</tr>
</tbody>
</table>
A parametric study of transmitter PWAS size effect of power attached to a simple aluminum plate. Numerical simulation was performed with the parameters given in Table 1. Constant 10-V excitation voltage from an ideal electrical source was assumed at the transmitter PWAS. The circular PWAS radius was varied from 2 to 15 mm, whereas the frequency was spanned from 1 to 1000 kHz.

Figure 6 presents the results of a parameter study of various PWAS sizes and frequencies. The resulting parameter plots are presented as 3D mesh plots. Figure 6a shows a 3D mesh plot of the power rating vs. frequency and transmitter size: for a certain circular transmitter radius, the power rating increases when the frequency increases for a given frequency,
the power rating increases when the transmitter size increases. These results demonstrate: to drive a 15-mm radius circular PWAS at 1000 kHz with a 10 V constant voltage input, one needs a power source providing 10 W of power. Figure 6b shows the wave power that a circular PWAS generates into the structure; tuning effect of transmitter radius and excitation frequency are apparent; a larger PWAS does not necessarily produce more wave power at a given frequency! The maximum wave power output in this simulation is \(\sim 20\text{mW}\). The wave power is the same as the electrical active power. This study provides guidelines for the design of transmitter size and excitation frequency in order to obtain maximum wave power into the SHM structure.

The powers contained in the axial waves and flexural waves are presented separately in Figure 6c and Figure 6d. In some PWAS SHM applications, a single mode is often desired to reduce signal complexity and simplify signal interpretation and damage detection. Figure 6c shows the frequency-size combinations at which the axial waves are maximized, whereas Figure 6d indicates the combinations that would maximize the flexural waves. These figures give useful guidelines for choosing PWAS size and frequency values those are optimum for selecting a certain excitation wave mode.

### 4. 2D CIRCULAR PWAS RECEIVER MODEL

The voltage response of a circular sensing PWAS is calculated by integrating the strain field over the entire area of the sensing PWAS and considering the piezoelectricity of the sensing PWAS.

\[
V_R = \frac{1}{Y_e + (1-k_p^2)Y_s} \int_0^{2\pi} \int_0^c \left( \varepsilon_{rr} + \varepsilon_{\theta\theta} \right) \kappa A \, dA
\]

where \(Y_e\) is external admittance and \(Y_s = i\omega C_s = i\omega \frac{E_{33}c^2}{h_{PWAS}}\) is the PWAS receiver admittance.

The strain can be written as

\[
\varepsilon_{rr} + \varepsilon_{\theta\theta} = \frac{\partial u_r}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} + \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}
\]

Bessel function has its property,

\[
\frac{\partial}{\partial r} J_j(\xi r) = \xi J_j(\xi r) - \frac{1}{r} J_{j+1}(\xi r)
\]

Recall the wave displacement equation and take the derivative, the strain becomes

\[
\varepsilon_{rr} + \varepsilon_{\theta\theta} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} = i\pi a e_0 \left[ J_0(\xi_0 a) J_0(\xi r) + 3J_1(\xi_0 a) J_0(\xi r) \right] e^{-iat}
\]

Recall equation (2.16) and \(c \ll r_s\), the integral of strain under the receiver area is

\[
\int_A \left( \varepsilon_{rr} + \varepsilon_{\theta\theta} \right) dA = i\pi a e_0 \int_A \left[ J_0(\xi_0 a) J_0(\xi r) + 3J_1(\xi_0 a) J_0(\xi r) \right] r dr d\theta
\]

\[
= 2\sqrt{\frac{2}{\pi \xi s}} \left[ J_1(\xi_0 a) \int_{r-c}^{r+c} \sqrt{\xi_0} \left( c^2 - (r_s - r)^2 \right) dr + 3J_1(\xi_0 a) \int_{r-c}^{r+c} \sqrt{\xi_0} \left( c^2 - (r_s - r)^2 \right) dr \right]
\]

This can be further simplified using the Poisson integral of the Bessel function. Finally the output voltage of the circular PWAS receiver with a radius of \(c\) corresponding to the PWAS transmitter with a radius of \(a\) becomes

\[
V_R(t) = -\frac{2c e0}{Y_e + (1-k_p^2)Y_s} \int_0^{2\pi} \int_0^c \left( J_1(\xi_0 a) J_1(\xi_0 c) e^{i(\xi_0 a - \xi r - \pi/4)} + 3 \frac{1}{\sqrt{\xi_{1/2}}} J_1(\xi_0 a) J_1(\xi_0 c) e^{i(\xi_0 a - \xi r - \pi/4)} \right) e^{-iat}
\]
\[ V_R = C^A (r_s) S^A (a, c) + C^F (r_s) S^F (a, c) \]

\[ C^A (r_s) = -2 \omega \sigma_0 \frac{d_{31}}{s_{11}} \sqrt{\frac{2 \pi}{\varepsilon_0 r_s}} e^{\frac{i (\varepsilon_0 r_s^2 - r_s^2)}{4}} e^{-i \omega t} \quad S^A (a, c) = \frac{\pi a c}{Y_e + (1 - k_p^2) Y_c} J_1 (\xi_0 a) J_1 (\xi_0 c) \]  
\[ C^F (r_s) = -6 \omega \sigma_0 \frac{d_{31}}{s_{11}} \sqrt{\frac{2 \pi}{\varepsilon_0 F r_s}} e^{\frac{i (\varepsilon_0 F r_s^2 - r_s^2)}{4}} e^{-i \omega t} \quad S^F (a, c) = \frac{\pi a c}{Y_e + (1 - k_p^2) Y_c} J_1 (\xi_F a) J_1 (\xi_F c) \]

The time-average electrical output power for receiver is

\[ \langle P(\omega) \rangle = \frac{1}{2} \text{Re} \left( V_R(\omega) I(\omega) \right) \quad \text{and} \quad I_R = Y_e V_R \]  

5. 2D PITCH-CATCH PWAS MODEL

The power and energy transduction flow chart for a complete pitch-catch setup is shown in Figure 7. Under 1-D assumption, the electro-acoustic power and energy transduction of the PWAS transmitter and receiver are examined. In pitch-catch mode, the power flow converts from electrical source into piezoelectric power at the transmitter, the piezoelectric transduction converts the electrical power into the mechanical interface power at the transmitter PWAS and then into acoustic wave power travelling in the structure. The wave power arrives at the receiver PWAS and is captured at the mechanical interface between the receiver PWAS at the structure, the mechanical power captured is converted back into electrical power in the receiver PWAS and captured at the receivers electric instrument. The time-averaged electrical power, mechanical power at the transmitter and wave power can be calculated from the frequency response function. The time-averaged mechanical power and electrical power at the receiver PWAS can be calculated as well.

5.1 Pitch-catch simulation

In pitch-catch setup simulation, an Aluminum alloy 2024 infinite plate with 1 mm thickness. PWAS transmitter and receiver are 7-mm diameter and 0.2-mm thickness. A 20-Vpp 100-kHz central frequency 3-count Hanning window tone-burst signal is applied to the transmitter. The receiver instantaneous voltage response is shown in. The fast axial wave is separated from the low speed flexural wave. The axial wave is non-dispersive and keeps the shape of excitation signal. The flexural wave spreads out due to the dispersive nature.
The receiver RMS power is defined as 
\[ RMS = \sqrt{\frac{\int_0^\tau V^2 \, dt}{\tau}} \], we can calculate the receiver RMS power (Figure 8b). It is clear that the receiver RMS power is proportional with \( \sqrt{1/r_s} \).

Figure 8 Pitch-catch signal with a receiver PWAS (a) at different distance from a transmitter PWAS, (b) RMS power of a receiver at different distance.

6. CONCLUSION AND FURTHER WORK

A systematic investigation of power and energy transduction in circular PWAS attached on structure was considered using the 2-D wave propagation methods. With 2-D wave propagation, ideal bonding and ideal excitation source assumption, frequency response function and power and energy transduction of PWAS transmitter and receiver were developed.

For a circular PWAS transmitter, the active power, reactive power, power rating of electrical requirement were determined under harmonic voltage excitation. It indicates that the reactive power is dominant and gives the power requirement for power supply / amplifier for PWAS application. The electrical and mechanical power analysis at the PWAS structure interface indicates all the active electrical power provides the mechanical power at the interface. This provides the power and energy for the axial and flexural wave’s power and energy travelling in the structure. The parametric study of PWAS transmitter size shows the proper size and excitation frequency selection based on the tuning effects.

For a circular PWAS receiver under a circular PWAS excitation, the frequency response function was also developed. The parametric study of the distance of the PWAS transmitter and receiver indicated the 2-D power spread out. The future work is the power and energy analysis with multi-mode Lamb waves.

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8. REFERENCES