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Technical Briefs

Statistical Analysis of the Dynamic Mean Excitation for a Spur Gear

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An innovative statistical methodology and generic expressions have been developed for the ratio of the dynamic mean excitation to the transmitted mean load under the influence of the important gear characteristics. Simple expressions have been established for identifying cases in which the dynamic mean excitation is more than, less than, or equal to the transmitted mean load. [DOI: 10.1115/1.1891812]

1 Introduction

The dynamic excitation generated in a gear influences its fatigue life and durability.

To date, several authors have outlined the relationships between dynamic excitation and gear parameters. Mark [1] studied the relationships between dynamic excitation and static transmission errors. Remmers [2] minimized the spectrum of the dynamic excitation for the expected load, speed range, and gear response. Umezawa et al. [3] investigated the influence of gear dimension parameters on the dynamic excitation for spur gears. Winter et al. [4] grouped the dynamic excitation of gears into three parts due to kinematical data, face width, and speed and studied the influence of the contact ratio, tooth variation, and profile modification on the first part. Kubo et al. [5–7] investigated the dynamic excitation of gears as a function of the gear errors, mesh stiffness, and dimensions.

Most of the studies focused on the influence of gear parameters on the total dynamic excitation (without making a distinction between the mean and alternating components) or on the alternating components of the dynamic excitation. The standard deviation of the dynamic excitation depends on the alternating component of dynamic excitation due to time-varying static transmission error and time-varying stiffness. Alternating components of dynamic excitation, including parametric excitation due to the time-varying mesh stiffness, are important for residual life estimation of gears.

However, fatigue life depends not only on the total dynamic load, but also on the ratio between the alternating and mean loads [8]. Though the *mean* of the dynamic excitation defines *the dynamic mean load* and therefore is highly important for practice, i.e., for fatigue life estimation of gears, it has seldom been investigated. In addition, most studies have used a deterministic approach without taking into account the statistical correlation between the mesh stiffness and static transmission error.

The purpose of this paper is to investigate the relationship between the dynamic *mean* excitation and the transmitted mean load.

2 Theoretical Analysis

A spur gear pair can be presented using two disks coupled by mesh stiffness and damping. One disk has the radius R_1 and the mass moment of inertia I_1 and the other the radius R_2 and the mass moment of inertia I_2. The equation for the dynamic transmission error x is as follows [2,9]:

$$mx'' + c[x - E(t)]' + K(t)[x - E(t)] = F$$

where $m=I_1I_2/(I_1R_2^2+I_2R_1^2)$ is the equivalent inertia mass; $x=R_1\theta_1-R_2\theta_2$; θ_1 and θ_2 are the gear angular displacements; $(\ )'=d/dt$; c is the equivalent damping coefficient; $E(t)$ is the equivalent time-varying gear error due to manufacturing errors (tooth flank deviation, pressure angle error, pitch error, run-out errors, etc.), exploitation errors (wear, etc), and gear profile modification; $K(t)$ is the equivalent time-varying mesh stiffness; $F=T_1/R_1=T_2/R_2$; T_1 and T_2 are the driving and driven torques; and F is the transmitted mean load.

The above equation after transformations can be presented in the form

$$mx'' + cx' + K(t)x = K(t)E_f(t) + cE'(t), \tag{1}$$

where $E_f(t)=E(t)+F/K(t)$; $E_f(t)$ is the equivalent time-varying static transmission error at the transmitted load F; the static transmission error $E_f(t)$ consists of the gear error $E(t)$ and the time-varying tooth deflection (i.e., $F/K(t)$) [10,11] under the transmitted load; $E_f(t)$ and $K(t)$ are processes with deterministic and random components [1,12,13] and random initial time; a time-varying behavior of these processes depends significantly on the tooth-number in contact during the mesh period [2,3,9,14,15]; and the static transmission error $E_f(t)$ is related to manufacturing errors through the gear error $E(t)$.

To nondimensionalize Eq. (1), we use the following transformations:

$$y = \frac{x}{x_0}, \quad t_1 = \omega_{nd}t, \tag{2}$$

where y is the normalized dynamic transmission error; x_0 is the dynamic transmission error under the characteristic mean load (i.e., baseload) F_0, i.e., $x_0=F_0/\bar{K}$; $\bar{K}$ is the average stiffness; t_1 is the nondimensional time; and ω_{nd} is the circular equivalent natural frequency of the gear pair, $\omega_{nd}=\sqrt{\bar{K}/m}$.

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The baseload is the transmitted mean load for which gear profile is modified. For example, for long relief profile modification the baseload is equal to the design transmitted mean load [16,17].

The nondimensional equation is obtained from Eqs. (1) and (2):

$$\ddot{y} + 2\zeta\dot{y} + k(t_1)y = k(t_1)e_f(t_1) + 2\zeta\dot{e}(t_1), \tag{3}$$

where $(\dot{\ }) = d/dt_1$; ζ is the damping ratio, $\zeta = c/2\sqrt{\overline{K}m}$; $k(t_1)$ is the normalized stiffness, $k(t_1) = K(t_1)/\overline{K}$; $e_f(t_1)$ is the normalized static transmission error, $e_f(t_1) = E_f(t_1)/x_0$; $e(t_1)$ is the normalized gear error; and $e(t) = E(t)/x_0$.

The dynamic mean excitation is the mean of the total dynamic excitation acting during the mesh cycle on the meshing pair. We take into account that the parametric excitation (i.e., excitation due to the time-varying mesh stiffness in the left-hand side of Eq. (1)) is essential [18] only on parametric resonance (i.e., at $0.5\omega_{nd}$). This excitation for the commonly used rectangular mesh stiffness defines [18] the damped periodic dynamic transmission error. This damped transmission error is decreasing in time in the "stable" region. Therefore, the dynamic mean excitation is the mean of the right side of Eq. (1) [5,19].

To estimate the relationship between the dynamic mean excitation and the transmitted mean load, we define the *mean* excitation ratio R:

$$R = L_d/L_t, \tag{4}$$

where L_d and L_t are the normalized dynamic mean excitation and the normalized transmitted mean load.

The difference between the mean excitation ratio and the dynamic load factor [20–22] is in the numerator: we use the dynamic *mean* excitation. The normalized dynamic mean excitation is the mean of the normalized total dynamic excitation, i.e., the mean of the right side of Eq. (3):

$$L_d = m_1(ke_f + 2\zeta\dot{e}) = m_1(k)m_1(e_f) + r_{ke}d_kd_e + 2\zeta m_1(\dot{e}), \tag{5}$$

where m_1 is a mean symbol; r_{ke} is the normalized cross-correlation coefficient between the processes $k(t_1)$ and $e_f(t_1)$; and d_k and d_e are the standard deviations of the processes $k(t_1)$ and $e_f(t_1)$.

The normalized transmitted mean load is the ratio of the transmitted mean load to the baseload:

$$L_t = \frac{F}{F_0} = a \tag{6}$$

Using Eqs. (5) and (6), the ratio R can be cast in the form

$$R = \frac{m_1(k)m_1(e_f) + r_{ke}d_kd_e + 2\zeta m_1(\dot{e})}{a} \tag{7}$$

Examining Eq. (7), we notice that the ratio R depends on the means and standard deviations of the normalized time-varying mesh stiffness and static transmission error, the ratio of the transmitted mean load to the characteristic mean load, the damping ratio, the mean of the derivative of the normalized gear error, and the normalized cross-correlation coefficient between mesh stiffness and the static transmission error. This coefficient characterizes the *statistical* dynamic interaction between these functions. Using the novel methodology [23], we can estimate this coefficient for arbitrary time-varying functions $k(t_1)$ and $e_f(t_1)$.

Equation (7) is innovative; it takes into account the significant gear functions: ratios a and ζ and deterministic and random components of the processes $k(t_1)$, $e_f(t_1)$, and $e(t_1)$. The factors that control the ratio R are the

- nonzero multiplication of the means of the normalized mesh stiffness and static transmission error
- cross-correlation coefficient between the normalized mesh stiffness and static transmission error
- mean of the derivative of the normalized gear error
- the normalized transmitted mean load

We utilize a commonly used [9,14,19,24–27] assumption:

$$K(t) = \overline{K} + K_v(t), \tag{8}$$

where $K_\nu(t)$ is the alternating part of the stiffness $K(t)$, $m_1(K_\nu) = 0$.

Inserting Eq. (8) into Eq. (7) yields

$$R = 1 + \frac{\Delta m_1(e_f) + r_{ke}d_kd_e + 2\zeta m_1(\dot{e})}{a} \tag{9}$$

where $\Delta m_1(e_f) = m_1(e_f) - a$; $\Delta m(e_{f_1})$ is the difference between the mean of the normalized static transmission error and the normalized mean tooth deflection under the transmitted load [2].

Note that for the ideal case when $e(t) = 0$), e.g., for errorless gears (precision gears without exploitation errors) and uncorrected gears (without profile modification), the mean excitation ratio is equal unity.

The variable term of Eq. (9),

$$V = \frac{\Delta m_1(e_f) + r_{ke}d_kd_e + 2\zeta m_1(\dot{e})}{a}, \tag{10}$$

is generally nonzero [9,14–16,24].

When term (10) is zero, the ratio R is unity, i.e., the dynamic mean excitation will equal the transmitted mean load. This situation will happen in the following cases:

- The factors in the numerator of (9) are so small or the factor in the denominator is so big that the term (10) can be neglected.
- The factors in the numerator are equal to zero.
- The factors in the numerator compensate for each other: $\Delta m_1(e_f) + r_{ke}d_kd_e + 2\zeta m_1(\dot{e}) = 0$.

When we neglect [24] the difference $\Delta m(e_{f_1})$ and the multiplication $2\zeta m_1(\dot{e})$, the ratio R takes the simpler form:

$$R = 1 + \frac{r_{ke}d_kd_e}{a} \tag{11}$$

Now, we analyze the ratio R for gears with the profile modification and long relief [16,17]. For these gears, the design mean load is used as the baseload. The deterministic normalized and dimensional static transmission errors (Harris maps) and mesh stiffness with nonzero means are presented in the classic papers [9,15] and in Refs. [2,14,16,17] for various transmitted loads. Considering these curves for various transmitted loads [2,12,15–17] and taking into account the number of tooth-pairs in contact during the mesh period, we notice that when the transmitted load is

- more than the design load, the mesh stiffness and the static transmission error are *off phase* and the normalized cross-correlation coefficient is less than zero:

$$r_{ke} < 0 \quad \text{if } a > 1 \tag{12}$$

- less than the design load, the Harris time histories of the static transmission error change the phase with respect to the number of tooth-pairs in contact; the mesh stiffness does not change the phase with respect to the number of tooth-pairs in contact; therefore the mesh stiffness and the static transmission error are *in phase* [9] and the cross-correlation coefficient is more than zero:

$$r_{ke} > 0 \quad \text{if } a < 1 \tag{13}$$

Using Eqs. (7), (12), and (13), and taking into account that $d_k > 0$, $d_e > 0$, we find that when the transmitted mean load

- is more than the design mean load, the cross correlation between the mesh stiffness and the static transmission error leads to a decrease in the ratio R
- is less than the design mean load, the cross correlation between the mesh stiffness and the static transmission error leads to an increase in the ratio R
- equals the design mean load, the cross-correlation coefficient, r_{ke}, the deviation, d_e, the difference, $\Delta m_1(e_f)$, and the mean of the derivative of the normalized gear error, $m_1(\dot{e})$, all equal zero, provided that ideal profile modification exists [2,16,17]; in this case, Eq. (7) yields $R=1$, e.g., the dynamic mean excitation equal to the transmitted mean load.

Using the simplified Eq. (11) we obtain that if the transmitted mean load is

- more than the design mean load, i.e., overload, then the inequality (12) is valid, and the dynamic mean excitation is less than the transmitted mean load
- less than the design mean load, i.e., underload, then the inequality (13) is valid and the dynamic mean excitation is more than the transmitted mean load

Thus, the underload produces a greater mean excitation ratio than the overload.

Equation (9) yields the following criteria:

- if $\Delta m_1(e_f)+r_{ke}d_kd_e+2\zeta m_1(\dot{e})>0$, then the dynamic mean excitation is more than the transmitted mean load
- if $\Delta m_1(e_f)+r_{ke}d_kd_e+2\zeta m_1(\dot{e})<0$, then the dynamic mean excitation is less than the transmitted mean load

These criteria can be used for influence analysis of the gear parameters on the dynamic mean excitation and the ratio R. Now we utilize a commonly used [14,19,24–26] assumption:

$$m_1(\dot{e}) = 0 \tag{14}$$

Inserting Eq. (14) into Eq. (9) yields the simpler form of the mean excitation ratio:

$$R = 1 + \frac{\Delta m_1(e_f) + r_{ke}d_kd_e}{a} \tag{15}$$

3 Examples

We have run two numerical examples using, without loss of the generality, deterministic components of the functions $K(t)$ and $E_f(t)$ and Eq. (8).

Case 1

- harmonic mesh stiffness [14,15,19,25,26]
- contact ratio $\varepsilon=1.5$
- $a=0.25$ and $a=0.75$

For this case and the harmonic gear error, the time variations of the normalized static transmission errors for long relief were presented in Ref. [15].

Using Eq. (2) and the mesh stiffness of Ref. [26], we identify the primary harmonic component of $k(t_1)$ in the form

$$k_1(t_1) = 2\sin(\varepsilon - n)\pi \cos \phi t_1/\pi\varepsilon, \tag{16}$$

where ϕ is the ratio of the meshing frequency to the natural frequency of the gear pair, n is an integer, $n\leqslant\varepsilon\leqslant n+1$. The variance of the stiffness (16) can be presented in the form

$$d_{kc}^2 = 2\sin^2(\varepsilon - n)\pi/(\pi\varepsilon)^2 \tag{17}$$

Using Eqs. (16) and (17), the normalized static transmission errors from Ref. [15], and work [28] we obtain, after transformations, $d_{kc}=0.3$, and

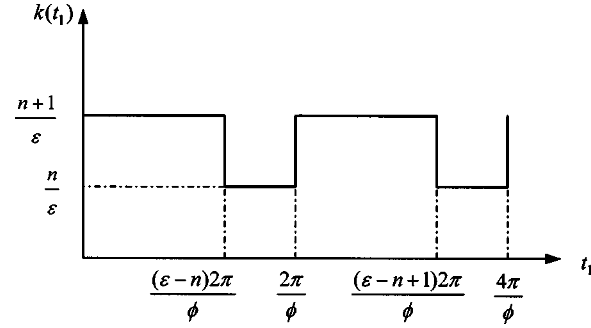

Fig. 1 Time variation of the mesh stiffness

- for $a=0.25$: $m_1(e_f)=1.05a$, $\Delta m_1(e_f)=0.05a$, $r_{ke}=1$, $d_e=0.55a$
- for $a=0.75$: $m_1(e_f)=1.04a$, $\Delta m_1(e_f)=0.04a$, $r_{ke}=0.5$, $d_e=0.05a$

Then, Eq. (15) yields $R=1.2$ and $R=1.05$ for $a=0.25$ and $a=0.75$, respectively. This indicates that the dynamic mean excitation is more than the transmitted load. In addition, we notice that the ratio R increases when the ratio a decreases in the range [1,0]. This is due to two tendencies which are valid for long relief: (a) the increase of the standard deviation of the static error [2,9,15–17] and (b) the increase of the cross-correlation coefficient with the decrease of the ratio a.

Hence, we conclude that, in this case, the main factors that make the ratio R depart from unity are

- the correlation between the time-varying mesh stiffness and the static transmission error for $a=0.25$
- the difference between the mean of the normalized static transmission error and the mean normalized tooth deflection under the transmitted load for $a=0.75$

Case 2

- rectangular mesh stiffness [9,26]
- contact ratio $\varepsilon=1.5$
- $a=1.5$

The normalized static transmission error is rectangular for long relief and "ideal profile modification" [2,16]. Using the rectangular mesh stiffness of Ref. [26] and Eq. (2), we can obtain, after transformations, the normalized mesh stiffness (Fig. 1). The variance of $k(t_1)$ can be written in the form

$$d_{kr}^2 = (\varepsilon - n)(n + 1 - \varepsilon)/\varepsilon \tag{18}$$

Using Eq. (18), the normalized stiffness (Fig. 1), the normalized static transmission errors from Ref. [2], and the work [28], we obtain $m_1(e_f)=0.92a$, $\Delta m_1(e_f)=-0.08a$, $r_{ke}=-1$, $d_{kr}=0.3$, and $d_e=0.08a$. With these values, Eq. (15) yields $R=0.9$, e.g., the dynamic mean excitation is less than the transmitted mean load. Hence, we conclude that in this case the main factor that makes the ratio R depart from unity is the difference between the mean of the normalized static transmission error and the mean normalized tooth deflection under transmitted load.

4 Conclusions

1. We have developed an innovative statistical methodology and generic expressions for the ratio of the dynamic mean excitation to the transmitted mean load under the influence of the following important gear characteristics:

- mean and the standard deviation of the mesh stiffness

- mean and the standard deviation the static transmission error
- mean of the transmitted load
- mean of the derivative of the gear error
- cross-correlation coefficient between the mesh stiffness and the static transmission error

This cross-correlation characterizes the statistical dynamic interaction between the mesh stiffness and the static transmission error. The difference between the dynamic mean excitation and the transmitted mean load is controlled by the following factors:

- the difference between the mean of the normalized static transmission error and the mean of the normalized tooth deflection under the transmitted load
- the cross correlation between the normalized mesh stiffness and the static transmission error
- the normalized transmitted mean load
- the mean of the derivative of the normalized gear error

2. Simple expressions have been established for identifying cases in which the dynamic mean excitation is more than, less than, or equal to the transmitted mean load.

3. For gears with the profile modification and long relief, it was found that when the transmitted mean load

- is more than the design mean load, the statistical correlation between the mesh stiffness and the static transmission error leads to a decrease in the mean excitation ratio (4)
- is less than the design mean load, the statistical correlation between the mesh stiffness and the static transmission error leads to an increase in the mean excitation ratio (4)
- equals the design mean load and ideal profile modification applies, the dynamic mean excitation equals the transmitted mean load

Thus, the underload produces a greater mean excitation ratio than the overload.

4. Numerical examples match with the theoretical results. We have shown that when the transmitted mean load is

- more than the design mean load, the dynamic mean excitation is less than the transmitted mean load
- less than the design mean load, the dynamic mean excitation is more than the transmitted mean load

It was shown that the dynamic mean excitation could be up to 20% more than 10% less than the transmitted mean load. These results are practically important for residual life estimation of gears.

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References

[1] Mark, W., 1978, "Analysis of the Vibratory Excitation of Gear Systems: Basic Theory," J. Acoust. Soc. Am., **63**(5), pp. 1409–1430.

[2] Remmers, E., 1978, "Gear Mesh Excitation Spectra for Arbitrary Tooth Spacing Errors, Load and Design Contact Ratio," ASME J. Mech. Des., **100**, pp. 715–722.

[3] Umezawa, K., Sato, T., and Kohno, K., 1984, "Influence of Gear Error on Rotational Vibration of Power Transmission Spur Gear," Bull. JSME, **27**(225), pp. 569–575.

[4] Winter, H., Gerber, H., and Muller, R., 1989, "Investigations on the Excitation of Vibrations and Noise at Spur and Helical Gears," Proceedings of the International Power Transmission and Gearing Conference, Chicago, IL, pp. 765–772.

[5] Kubo, A., and Kiyono, S., 1980, "Vibrational Excitation of Cylindrical Involute Gears Due to Tooth Form Error," Bull. JSME, **23**(183), pp. 1536–1543.

[6] Kubo, A., and Kawasaki, Y., 1992, "Silent Gear Design for Mass-Produced Gears with Scatters in Tooth Form Accuracy," Proceedings of the International Power Transmission and Gearing Conference, Phoenix, AZ, pp. 589–595.

[7] Kubo, A., Kawasaki, Y., and Komori, M., 2000, "Design Method of Vibrationally Optimum Flank Form for Involute Helical Gears with Scattering in Pressure Angle and Helix Angle Deviation," Proceedings of the International Power Transmission and Gearing Conference, Baltimore, MD, pp. 529–537.

[8] Kaplan, M., and Wolff, W., 1996, "Life Extension and Damage Tolerance of Aircraft," ASM Handbook, **19**, pp. 557–565.

[9] Harris, S., 1958, "Dynamic Loads on the Teeth of Spur Gears," Proc. Inst. Mech. Eng., **172**, pp. 87–112.

[10] Parker, R., Vijayakar, S., and Imajo, T., 2000, "Nonlinear Dynamic Response of a Spur Gear Pair: Modeling and Experimental Comparisons," J. Sound Vib., **237**, pp. 435–455.

[11] Remmers, E., 1972, "Analytical Gear Tooth Profile Design," ASME paper no. 72-PTG-47.

[12] Neriya, S., Bhat, R., and Sankar, T., 1988, "On the Dynamic Response of a Helical Geared System Subjected to a Static Transmission Error in the Form of Deterministic and Filtered White Noise Inputs," ASME J. Vib., Acoust., Stress, Reliab. Des., **110**, pp. 501–506.

[13] Sato, T., 1986, "Approximate Solution of a Gear Subject to Random Excitation," Bull. JSME, **29**(251), pp. 1586–1592.

[14] Kahraman, A., and Singh, R., 1991, "Interactions Between Time-Varying Mesh Stiffness and Clearance Non-Linearities in a Geared System," J. Sound Vib., **146**(1), pp. 135–156.

[15] Gregory, R., Harris, S., and Munro, R., 1963–1964, "Dynamic Behaviour of Spur Gears," Proc. Inst. Mech. Eng., **178**, pp. 207–226.

[16] Munro, R., 1989, "The D.C. Component of Gear Transmission Error," Proceedings of the International Power Transmission and Gearing Conference, Chicago, IL, pp. 467–470.

[17] Smith, J., 1999, *Gear Noise and Vibration*, Marcel Dekker, New York.

[18] Den-Hartog, J., 1985, *Mechanical Vibrations*, Dover, New York.

[19] Amabili, M., and Rivola, A., 1997, "Dynamic Analysis of Spur Gear Pairs: Steady-State Response and Stability of the SDOF Model with Time-Varying Mashing Damping," Mech. Syst. Signal Process., **11**(3), pp. 375–390.

[20] Seireg, A., and Houser, D., 1970, "Evaluation of Dynamic Factors for Spur and Helical Gears," ASME J. Eng. Ind., **1**, pp. 504–515.

[21] Liou, C., Lin, H., Oswald, F., and Townsend, P., 1996, "Effect of Contact Ratio on Spur Gear Dynamic Load With No Tooth Profile Modifications," ASME J. Mech. Des., **118**, pp. 439–443.

[22] Litvin, F., 1968, *Theory of Gearing*, Nauka, Moscow.

[23] Gelman, L., Burau, Y., and Krytsyn, V., 1995, "Automatic Adaptive Method for Testing Data Processing for Nondestructive Testing in Civil Engineering," Proceedings of the International Symposium of Nondestructive Testing in Civil Engineering, Berlin, Germany, Vol. 2, pp. 1311–1317.

[24] Kubo, A., Kiyono, S., and Fujino, M., 1986, "On Analysis and Prediction of Machine Vibration Caused by Gear Meshing," Bull. JSME, **29**(258), pp. 4424–4429.

[25] Blankenship, W., and Kahraman, A., 1996, "Gear Dynamics Experiments, Part-I: Characterization of Forced Response," Proceedings of the 7th International Power Transmission and Gearing Conference, San Diego, CA, pp. 373–380.

[26] Kahraman, A., and Blankenship, W., 1999, "Effect of Involute Contact Ratio on Spur Gear Dynamics," ASME J. Mech. Des., **121**, pp. 112–118.

[27] Kucukay, F., 1984, "Dynamic Behavior of High Speed Gears," Proceedings of the Third International Conference on Vibrations in Rotating Machinery, Yorkshire, pp. 81–90.

[28] Devore, J., 2000, *Probability and Statistics for Engineering and the Sciences*, Duxbury.