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Development of Strength Theories for Random Fiber Composites

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ABSTRACT: A reassessment of existing theories for calculating the strength of random and quasi-random fiber composites is presented. Fundamental aspects regarding the physical model, macromechanics analysis, fiber distribution functions, generalized failure criterion, and progressive *versus* sudden failure models are covered first. Progressive ductile failure, progressive brittle failure, and sudden brittle failure are treated in detail. In each case, the original theory is briefly reviewed, and then its extensions accompanied by numerical examples are presented. Several limitations originally imposed by Hahn, such as the monotonically nonincreasing requirement on the failure strain curve, are lifted and the mathematical formulations are generalized. Some common misconceptions are also highlighted and clarified. Comparison with experimental data is given for the SMC-R50 material system. Good reproduction of the experimental results and of the stress-strain response are illustrated. A review of the main points and opportunities for further work are presented in conclusion.

KEYWORDS: composites, strength, random orientation fibers, stress-strain response, effective properties

Nomenclature

Latin

$E(\theta)$	modulus of θ-ply
E_o	effective modulus of the laminate
E_x	on-axis longitudinal modulus
E_y	on-axis transverse modulus
$f_x(\theta)$, $f_y(\theta)$, $f_s(\theta)$	failure stress functions for longitudinal, transverse, and shear modes (maximum stress criterion)
$f_{Xe}(\theta)$, $f_{Ye}(\theta)$	variants of failure stress functions to be used with the maximum strain criterion
F^*_{xy}	interaction coefficient in a quadratic failure criterion
G_{xy}	on-axis shear modulus
$h(\theta)$	fiber orientation density function
$H(t)$	Heaviside function
k	constant coefficient in the *KR* orientation density function
I	θ-interval within $(0, \pi)$ where failure has occurred
S	on-axis shear strength
S_e	on-axis shear failure strain
t	generic variable in Heaviside and Dirac functions
X, X_t, X_c	on-axis longitudinal strengths
X_e, X_{te}, X_{ce}	on-axis longitudinal failure strains
$X(\theta)$	failure stress function in terms of θ
$\bar{X}$	ultimate strength of the laminate
Y_t, Y_c	on-axis transverse strengths
Y_{te}, Y_{ce}	on-axis transverse failure strains

Greek

α	ration of shear to transverse strengths
α_0, α_1, α_2	constant coefficients in the *W* orientation density function
$\delta(t)$	Dirac's function
ϵ	engineering strain
$\epsilon_0(\theta)$	failure strain function in terms of θ
θ	off-axis fiber orientation
θ_1	intersection of longitudinal and shear failure modes
θ_2	intersection of shear and transverse failure modes
θ_3	angle defining the minimum value of the shear failure strain function
θ_x, θ_y	critical angles in the maximum strain failure criterion
ν_{xy}, ν_{yx}	major and minor Poisson's ratios
σ_x, σ_y	on-axis longitudinal and transverse direct stresses
σ_s	on-axis shear stress

Subscripts

c	compression
e	strain
H	Hahn's uniform density function [2]
KR	Kenneth Reifsnider's density function
L	lowest value of $\epsilon_0(\theta)$
M	maximum value of the σ–ϵ curve

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W Wang's density function [7]
x, y, s on-axis longitudinal, transverse, and shear, respectively

Additional symbols and notations may be given in the text.

Introduction

The failure of composite materials has generated a great deal of interest, but so far a complete, comprehensive, generally applicable model for failure prediction has not been constructed. Even restricting our attention to filamentary composites, it can still be said that a good deal of research has to be done before a foolproof model can be achieved.

First strength predictions are traced back to the classical works of Tsai [1] in the 1960s. Much research was done in the 1970s. Wide theoretical ground was covered, but the numerical applications were sometimes restricted by the inadequate computational technology of the period. To overcome such difficulties, simplifying (and some times oversimplifying) assumptions had to be made. One typical example is the dislike many researchers have acquired of the maximum stress criterion for its nonconvex nature. However, with present day computational technology, and the advent of powerful mathematical software (e.g., MATHEMATICA, MATHCAD, MATHLAB, etc.) some of these limitations can now be overcome. By releasing some of the artificial restrictions imposed on the mathematical treatment, extensions of existing theories can be obtained. Several extensions to Hahn's theory [2] for random fiber composites will be discussed here. A general mathematical formulation will also be included. A nonconvex strength criterion will be presented and used with mathematical ease. Comparison with previously reported numerical results will be made.

Fundamental Aspects

The Physical Model and the Macromechanics Analysis

According to Hahn [2] and other authors [3–5] a two-dimensional (2–D) random composite can be assumed to consist of a large number of superposed unidirectional "layers." Each layer is fully characterized by its orientation θ. The layered behavior is fully additive. In the limit, as the layer thicknesses become infinitesimally small, and the number of layers becomes infinitely large, a Lipschitz continuous variation is obtained and the usual summations can be replaced by integrals. Hence the initial (small load) elastic modulus is given by the continuum equivalent of the "rule of mixtures"

$$E_o = \int_{\theta_{min}}^{\theta_{max}} h(\theta)E(\theta)d\theta \tag{1}$$

where $E(\theta)$ is the one-dimensional (1–D) elastic modulus of the θ-ply, and $h(\theta)$ is a normalized density function for the ply orientation. The limits θ_{min}, θ_{max} define the range of θ under consideration, e.g., $(-\pi/2, \pi/2)$ for a general density function, or $(0, \pi/2)$ for a symmetrical density function.

Orthotropic elastic properties (E_x, E_y, ν_{xy}, and G_{xy}) for the basic ply with $\theta = 0°$ orientation are assumed. Using 1–D macromechanics analysis of layered composites [6], one gets the off-axis lamina elastic modulus as

$$E(\theta) = \left[\frac{1}{E_x}\cos^4\theta + \frac{1}{E_y}\sin^4\theta + \left(\frac{1}{G_{xy}} - 2\frac{\nu_{xy}}{E_x}\right)\cos^2\theta\sin^2\theta\right]^{-1} \tag{2}$$

The reference direction is the loading direction, and the orientation density function $h(\theta)$ is assumed to be symmetrical about the reference direction.

The Fiber Orientation Density Functions

It is assumed that the laminate consists of a large number of thin plies with various orientations θ with respect to the loading direction, Ox. Each ply is uniform and consists of parallel unidirectional fibers. The fraction of plies having angle orientation θ is given by the orientation density function $h(\theta)$ obeying the normalization relation

$$\int_{\theta_{min}}^{\theta_{max}} h(\theta)d\theta = 1 \tag{3}$$

The angles θ_{min} and θ_{max} are usually 0 and $\pi/2$ for a θ-symmetric laminate, but other ranges of θ can also be considered.

In his original work [2], Hahn used a uniform orientation density function defined over the range $(0, \pi/2)$ and given by

$$h_H(\theta) = \frac{2}{\pi} \tag{4}$$

It is, however, conceivable that nonuniform orientation density functions can be encountered in practice. Several reasons should be considered. One would be that of possible deficiency in the fabrication process, such that a nonuniform orientation density is obtained over the θ-domain. Another would be that of intentionally biased fiber orientation such that a preferred directionality is obtained in the final composite. (Such biases could correspond to known directions of maximum loads.) The fabrication process of random composites (e.g., SMC) can be easily modified to obtain such a bias; for example, if the fibers were laid down with a blower, the nozzle shape and nozzle movements can be adjusted and an orientation density having a preferred direction can be obtained. A third case would that of a "Flow Mat" type composite that would produce fiber orientations biased towards a specific direction according to some viscous fluid dynamics patterns during the injection process. Hence, whatever the cause, it is important to study composites with a nonuniform, or biased, quasi-random fiber orientation.

Two nonuniform orientation density functions were studied. One is a simple decaying exponential

$$h_{KR}(\theta) = ke^{(-2\theta)}, \quad 0 < \theta < \frac{\pi}{2} \tag{5}$$

where k is an arbitrary constant to be fixed through the normalization process (Eq 3). This function was proposed by the second author based on experience and insight. The other nonuniform

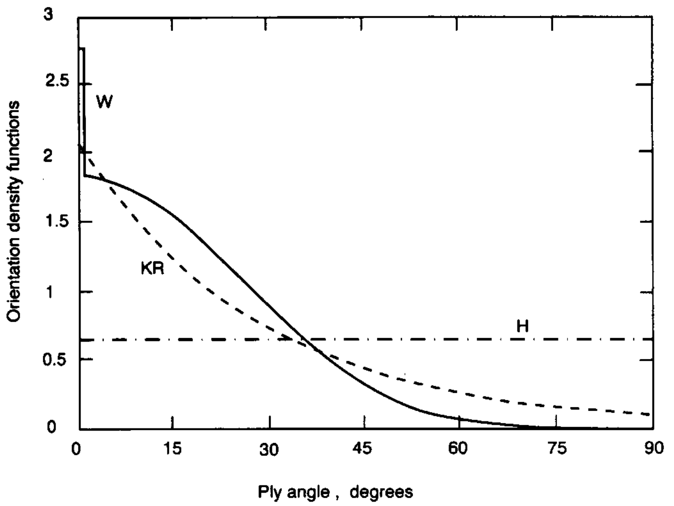

FIG. 1—*Various random and quasi-random fiber orientation density functions:* H = *uniform;* KR = *decaying exponential;* W = *biased trigonometric.*

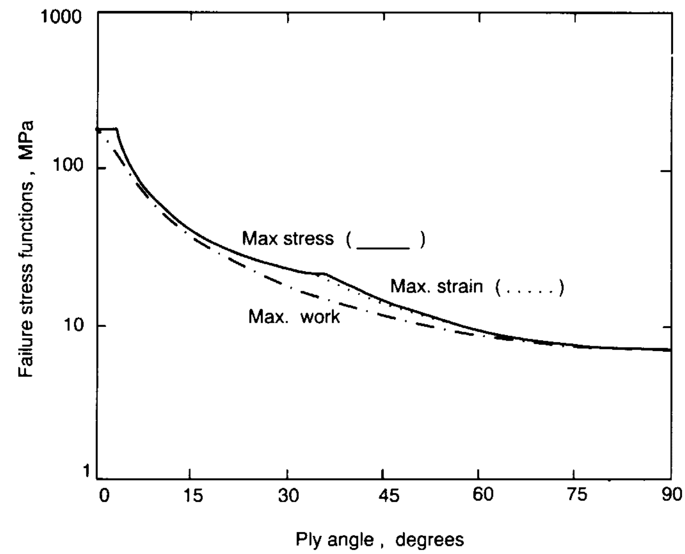

FIG. 2—*Failure stress functions for maximum stress, maximum strain, and maximum work criteria* (X_t = *1750 MPa,* Y_t = *70 MPa,* S = *100 MPa).*

orientation density is a more complicated law containing a mixture of trigonometric functions, and also a Dirac function to take account of an additional peak along the preferred direction

$$h_w(\theta) = \alpha_o\delta(\theta) + (\alpha_1 \cos^4\theta + \alpha_2 \cos^2\theta \sin^2\theta) \cos\theta, \quad 0 < \theta < \frac{\pi}{2} \tag{6}$$

This function was originally proposed by S. S. Wang [*7*] to describe the orientation density of cracks in a fatigued random composite, but we believe that its generality is sufficient to allow us to use it in the present situation.

Thus, besides the simple uniform orientation density used by Hahn [*2*] for simulating an ideally random composite, we also tested other distributions, simulating nonuniform, and even biased, quasi-random composites (Fig. 1). This allowed us to compare the effect of nonuniform fiber distributions, which might occur either purposefully or accidentally, and to comment on its consequences.

The Generalized Failure Function

Assume that the basic ply strengths in the fiber direction (X_t, X_c), transverse direction (Y_t, Y_c) and shear (S), as well as the failure strains (X_{te}, X_{ce}), (Y_{te}, Y_{ce}) and S_e, are known from independent on-axis tests.

In his 1975 paper, Hahn [*2*] used the maximum stress failure criterion. However, as early as 1966, Tsai [*8*] pointed out the basic differences between maximum stress, maximum strain, and maximum work failure criteria. Tsai's preference was towards using the maximum work criterion

$$\frac{\sigma_x^2}{X^2} - \frac{\sigma_x}{X} \cdot \frac{\sigma_y}{Y} + \frac{\sigma_y^2}{Y^2} + \frac{\sigma_s^2}{S^2} = 1 \tag{7}$$

due to a number of beneficial features that make it easier for numerical manipulation. Some of these features are: continuous variation with θ in an off-axis test; monotonically nonincreasing behavior as a function of θ; and account of failure mode interaction. However, we would like to point out that the first two features are specific to any quadratic function formulation, and also that, to date, a multitude of proposals exist for an interaction term F_{xy}^*, leading to many different quadratic failure criteria of the general form

$$\frac{\sigma_x^2}{X_tX_s} + 2F_{xy}^* \frac{\sigma_x\sigma_y}{\sqrt{X_tX_cY_tY_c}} + \frac{\sigma_Y^2}{Y_tY_s} + \frac{\sigma_s^2}{S^2} + \left(\frac{1}{X_t} - \frac{1}{X_s}\right)\sigma_x + \left(\frac{1}{Y_t} - \frac{1}{Y_s}\right)\sigma_y = 1 \tag{8}$$

On the other hand, the maximum stress

$$-X_c < \sigma_x < X_t \qquad -Y_c < \sigma_y < Y_t \qquad -S < \sigma_s < S \tag{9}$$

and the maximum strain

$$-X_{ce} < \epsilon_x < X_{te} \qquad -Y_{ce} < \epsilon_y < Y_{ce} \qquad -S_e < \epsilon_s < S_e \tag{10}$$

failure criteria are based on inequalities and physical insight. They have a discontinuous formulation and do not ensure monotonically nonincreasing behavior. But with modern computing currently available on even the low-range PCs, these features are no longer a handicap. And, there is still no clear-cut evidence in the available experimental data to recommend (with total certainty) the exclusive use of a particular type of failure criterion. The clearly traceable physical insight is a major advantage of the maximum stress and maximum strain criteria. A comparative plot of the various criteria applicable for a simple tension test of an off-axis composite is presented in Fig. 2.

Based on the maximum stress criteria, we express the general form of the failure stress function $X(\theta)$ for unidirectional tension loading of a generic ply with off-axis orientation θ as

$$X(\theta) = \min\left(f_x(\theta), f_y(\theta), f_s(\theta)\right) \tag{11}$$

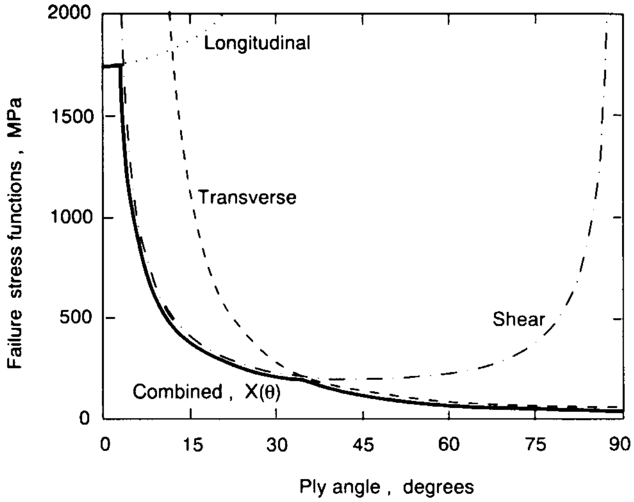

FIG. 3—*Interaction of the three failure mechanisms* (X_t = *1750 MPa*, Y_t = *70 MPa*, S = *100 MPa*).

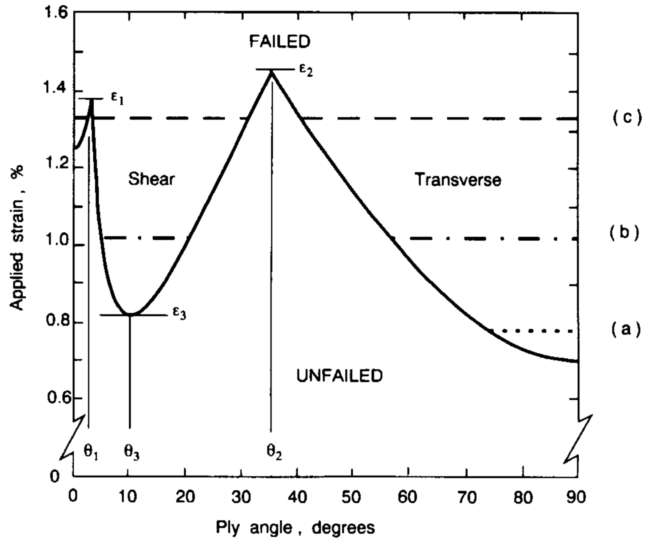

FIG. 4—*Failure strain function for maximum stress criterion* (X_t = *1750 MPa*, Y_t = *70 MPa*, S = *100 MPa*, E_x = *140 GPa*, E_y = *10 GPa*, G_{xy} = *4 GPa*, ν_{xy} = *0.31*).

where

$$f_x(\theta) = \frac{X}{\cos^2\theta},\quad f_Y(\theta) = \frac{Y}{\sin^2\theta},\quad f_s(\theta) = \frac{S}{\cos\theta\,\sin\theta} \tag{12}$$

are the failure functions for each failure mechanism, i.e., longitudinal to the fibers, transverse to the fibers, and shear. They represent the value of the loading stress that will induce failure in an uniform laminate with fiber orientation θ according to each of the above mechanisms. Though one cannot, *a priori*, know which of the failure mechanisms will occur, it is reasonable to assume that the ply will "yield" under the mechanism for which the failure stress is reached first. Hence, the strength of the ply at various orientations, θ, is given by Eq 11. Figure 3 illustrates the three strength curves for a ply in the range $0 < \theta < \pi/2$. It can be seen that the effective strength according to this criterion is the lower envelope of the three basic curves.

Associated with the strength function $X(\theta)$ one can draw the failure strain function $\epsilon_0(\theta)$, as shown in Fig. 4, and defined by

$$\epsilon_o(\theta) = X(\theta)/E(\theta) \tag{13}$$

where $E(\theta)$ is the tensile modulus (Eq 2) of the θ-ply in the applied stress direction. The fact that the strength $f_x(\theta)$ increases with θ, while the effective modulus $E(\theta)$ decreases with θ, leads to an interesting phenomenon (Fig. 4): According to this criterion, the failure strain for a slightly off-axis ply is larger than the failure strain for the perfectly oriented lamina. However, this beneficial aspect cannot be extended, since a further increase in θ produces a switch of the failure mechanism from fiber fracture into matrix shear. This latter mechanism has a fast rate of strength decrease with the off-axis orientation θ.

Figure 4 also shows that a random fiber composite subjected to increasing strain ε, will start to fail at an early stage. The 90° "layers" may experience failure first as the applied strain exceeds the transverse failure strain, and for a while failure proceeds towards smaller and smaller θ's (e.g, Level (a) in Fig. 4). However, this soon changes if the failure strain curve does not present a monotonically nonincreasing behavior. Figure 4 shows that for Level (b) two disjoint failure regions exist simultaneously: one inside the shear failure zone and the other inside the transverse failure zone. As the applied strain is increased further, a three-region situation may occur (Level (c) in Fig. 4): one of shear-, one of transverse-, and one of tensile-failure.

As the corresponding peaks ϵ_1 and ϵ_2 are passed, these regions may join together. It is interesting to note that the influence of relative stiffness and strength properties in the basic ply may yield $\epsilon_2 > \epsilon_1$, i.e., the final failure may occur in shear.

For the maximum strain criterion we write in a similar manner

$$X_e(\theta) = \min\,(f_{Xe}(\theta), f_{Ye}(\theta), f_S(\theta)) \tag{14}$$

where

$$f_{Xe}(\theta) = \frac{X_e}{\cos^2\theta - \nu_x\sin^2\theta},\quad f_{Ye}(\theta) = \frac{Y_e}{\sin^2\theta - \nu_y\cos\theta} \tag{15}$$

Care must be given in using the appropriate branch of the $f_{Xe}(\theta)$ and $f_{Ye}(\theta)$ functions [9], as discussed in the Appendix.

The failure functions can thus be defined and handled by PC software as any other continuous mathematical function. Additionally, as more physical insight is gained, and other failure mechanisms are identified, the generalized maximum stress (or strain) failure function can easily be modified by adding new elements to the parameter list of the *minimum* functions $X(\theta)$ and $X_e(\theta)$.

The ratio, α, between transverse and shear strengths, is a controlling parameter for the shape of the aggregate failure function $X(\theta)$. In his original work [2], Hahn made extensive effort to differentiate between the functional behaviors corresponding to various values of α and to find experimental evidence regarding the actual range of α values. However, we will show in our examples that using the generalized mathematical formulation described above renders such efforts unnecessary. Work performed on various composite systems, with different ratios between the fiber direction, transverse direction, and shear strengths, proves this statement.

Progressive versus Sudden Failure Models

Progressive failure models assume that the laminate stiffness decreases progressively as more and more plies reach their "yield" strain and diminish their contribution to sustaining the applied load. The sudden failure model assumes that, once initiated, the failure propagates rapidly through the complete laminate. The post-yield behavior of the failed plies plays a major role, and three different mechanisms can be depicted:

1. *Progressive ductile failure.* In most cases failure of the ply will not necessarily lead to the complete loss of its load bearing capacity. A residual load bearing capacity will still be present, and a common assumption is that the failed ply will continue to support the value of stress attained when failure occurred, i.e., the "yield stress." (We note that this mechanism resembles the strain-hardening plastic behavior of certain metals.) Thus, a gradual decrease in the stress-strain slope will occur, and the ultimate failure happens when all the plies have yielded, and no more load increase can be attained (i.e., zero tangent modulus). The physical argument behind such behavior ties in with many observed phenomena, such as post-fracture fiber pull-out, the interlaminar sliding of fractured plies, yielding in metal matrix systems, etc. A possible micromechanical model for such occurrences is that of Coulomb friction. For a more precise analysis of the post-failure stress-strain behavior one can assume that only a fraction of the yield stress is retained, thus using a post-failure efficiency factor, k_y. This assumption could be tied in with the "static" and "kinematic" friction coefficients present in the detailed analysis of the Coulomb friction phenomenon.

2. *Progressive brittle failure.* It is assumed that ply failure is comprehensive, and leads to the total loss of its load carrying capacity. Hahn [*2*] attributes this situation to a weak matrix/weak interface system. Hence, after the θ-ply has failed, the remaining plies will have to take on an increased load. This is achieved through a load redistribution mechanism. The remaining plies will experience an increased strain (if the load is kept constant as in a soft machine), or a decreased stress (if the displacement is kept constant as in a stiff machine). An intermediate behavior is also possible [*10*]. Under this assumption, the ultimate failure occurs when an increased load can no longer be sustained through the load redistribution mechanism. Note that this assumption does not imply the failure of the strongest ply, and hence the maximum strength of the laminate could occur before failure of the strongest ply.

3. *The sudden brittle failure.* For brittle matrix composites it is possible that the failure of the weakest ply propagates across the laminate thickness and produces ultimate failure. Under this assumption, the laminate failure corresponds to the first failure occurring in the weakest ply.

The choice between the three failure mechanisms presented above can only be made with significant physical insight. For most polymeric composites, on of the two progressive phenomena occur, thus rending some degree of "advanced warning" in the failure process. An extensive analysis based on the more conservative progressive brittle failure mechanism was recently reported by Thomas and Wetherhold [*11*].

The Benchmark Experiment for Comparing Failure Models

The analysis of a typical laminate was conducted in a manner simulating a simple uniaxial test, with the laminate subjected to increasing load in a stiff machine (i.e., under controlled strain loading). Though the axial strain is the controlling parameter, the specimen is assumed to be in a state of plane stress under uniaxial tension, i.e., free to deform transversely according to Poisson effects. This is achieved by assuming that the boundary constraints at the specimen ends do not influence the waisted midsection area of the specimen. This assumption is consistent with most experimental data available in the literature [*1,12,13*] and with Hahn's [*2*] theoretical work.

Computational Methods

Various algorithms for numerically modeling the nonlinear failure behavior were tested, such as series summations, incremental load-displacement analysis, and closed form solutions. After initial trial tests, it was found that the best and fastest results could be obtained with closed form analytical expressions handled on typical PC software.

Progressive Ductile Failure

Review of Hahn's Model for Progressive Ductile Failure

Ductile behavior of the fibers, matrix, and interface is assumed. After yield has occurred, the ply continues to sustain the maximum stress attained just before yield (the "yield stress"). Such a behavior is typical of tough matrix failure (in shear or in transverse loading), but also can be assumed to occur in the fiber direction if the fiber-matrix interface is at least as strong as the fiber, and if post-yield fiber pull-out happens under comparatively high sustained stress.

Hence, one has to consider the incremental behavior of the tangent modulus $E_t(\theta)$ as given by

$$E_t(\epsilon) = E_o - \int_{\theta_{min}}^{\theta_{max}} h(\theta)E(\theta)H(\epsilon - \epsilon_o(\theta))d\theta \qquad (16)$$

where $H(t)$ is the Heaviside function for a generic variable t, and E_0 is the initial average modulus given by

$$E_o = \int_{\theta_{min}}^{\theta_{max}} h(\theta)E(\theta)d\theta \qquad (17)$$

At any load level ϵ the corresponding stress is

$$\sigma(\epsilon) = \int_o^{\epsilon} E_t(\varepsilon)d\varepsilon \qquad (18)$$

In his original work [*2*], Hahn rewrites equation (18) as

$$\sigma(\epsilon) = E_o\epsilon - \int_I h(\theta)E(\theta)(\epsilon - \epsilon_o(\theta))d\theta \qquad (19)$$

where the interval I consists of those angles for which $\epsilon_o(\theta) < \epsilon$. Hence, the ultimate strength is reached when all the plies have yielded and I has stretched over the whole θ-range, e.g., (0, π/2) in Hahn's work. Rearranging Eq 19 one gets the laminate strength as

$$\bar{X} = \int_o^{\pi/2} h(\theta)E(\theta)\epsilon_o(\theta)d\theta = \int_o^{\pi/2} h(\theta)X(\theta)d\theta \qquad (20)$$

which is a rule of mixtures equation for strength.

Hahn continues his development for a uniform random fiber composite and writes

$$\bar{X} = \frac{2}{\pi}\int_o^{\pi/2} X(\theta)d\theta \qquad (21)$$

Using the maximum stress failure criteria Hahn develops the expression

$$\bar{X} = \frac{2}{\pi}\left[X_t\int_o^{\theta_1}\frac{d\theta}{\cos^2\theta} + S\int_{\theta_1}^{\theta_2}\frac{d\theta}{\sin\theta\cos\theta} + Y_t\int_{\theta_2}^{\pi/2}\frac{d\theta}{\sin^2\theta}\right] \qquad (22)$$

where θ_1 and θ_2 are the intersection angles between various failure mechanisms, i.e.,

$$\theta_1 = \tan^{-1}(S/X_t), \quad \theta_2 = \tan^{-1}(Y_t/S) \qquad (23)$$

Developing Lees's [*14*] initial work, Hahn [*2*] wrote closed form solutions for the above integrals, according to the different values of the parameter $\alpha = S/Y$, i.e., whether transverse or shear failure mechanisms prevail first. However, for most current composite systems $\alpha < \sqrt{X_t/Y_t}$ and hence the usual failure mechanism transition sequence (longitudinal—shear—transverse) takes place as θ increases from 0 to $\pi/2$.

Extension of Hahn's Model for Progressive Ductile Failure

We first notice that the integral over the interval I (defined by Hahn as consisting of the angles at which $\epsilon_o(\theta) < \epsilon$) can be generalized using Heaviside's function $H(t)$ and hence Eq 19 can be rewritten as

$$\sigma(\epsilon) = E_o\epsilon - \int_o^{\pi/2} h(\theta)E(\theta)(\epsilon - \epsilon_o(\theta))H(\epsilon - \epsilon_o(\theta))d\theta \qquad (24)$$

thus substituting the variable interval I with the fixed domain $(0, \pi/2)$. Second, we notice that Eq 24 results from a double integral, i.e.,

$$\sigma(\epsilon) = E_o\epsilon - \int_o^{\epsilon}\int_o^{\pi/2} h(\theta)E(\theta)H(\epsilon - \epsilon_o(\theta))d\theta\, d\epsilon \qquad (25)$$

Rearranging the double integral and using the direct and complementary integration domains we can write

$$\sigma(\epsilon) = E_o\epsilon - \int_o^{\epsilon}\int_o^{\pi/2} h(\theta)E(\theta)d\theta\, d\epsilon + \iint_{A_2} h(\theta)E(\theta)d\theta\, d\epsilon \qquad (26)$$

where A_2 is the integration domain contained in the rectangular domain $(0, \pi) \times (0, \epsilon)$ and defined as

$$A_2 = \{(\theta, \epsilon) \mid \theta \in (0, \pi/2), \epsilon \in [0, \min(\epsilon, \epsilon_o(\theta))]\} \qquad (27)$$

One observes that the first and second terms in Eq 26 cancel out and hence the expression for $\sigma(\epsilon)$ greatly simplifies. In the most general case, with angle variations within $(\theta_{min}, \theta_{max})$, one writes

$$\sigma(\epsilon) = \int_{\theta_{min}}^{\theta_{max}} h(\theta)E(\theta)\min(\epsilon, \epsilon_o(\theta))d\theta \qquad (28)$$

This general expression is our extension to Hahn's ductile progressive failure model for random composites and, together with Eq 20 for ultimate strength, forms a comprehensive tool for the analysis of this type of failure behavior. This closed-form formulation is easy to handle with existing PC codes, and has several advantages over both discrete ply analysis and a step-by-step incremental approach. Several numerical examples with various combinations of fiber distributions, basic ply strengths, and failure criteria will be presented next.

The Influence of Fiber Orientation on the Progressive Ductile Failure Strength

The stress-strain curve for the progressive ductile failure of a uniformly random composite is given in Fig. 5, Curve H. It can be seen that the ultimate strength in this case is

$$\bar{X}_H = 287 \text{ MPa} \qquad (29)$$

Upon using the *KR* orientation density describing a nonuniform random fiber composite biased towards the 0° orientation one gets the stress-strain curve *KR* of Fig. 5, and the ultimate strength

$$\bar{X}_{KR} = 567 \text{ MPa} \qquad (30)$$

Finally, the strongly biased orientation density *W*, having 2% of the fibers purposefully oriented in the 0° direction, yields the *W* stress-strain law of Fig. 5, and the ultimate strength value

$$\bar{X}_w = 619 \text{ MPa} \qquad (31)$$

It is worth mentioning that based on our failure criterion all three stress-strain curves have the same ultimate strain of 1.45%; hence the increased strength is entirely due to the increased modulus. The above three cases show that considerable strength gain (117%) can be obtained in this type of failure using biased distributions, and the differential increase between the two nonuniform distributions under consideration (*KR* and *W*) is small, but significant (9%).

Progressive Brittle Failure

Review of Hahn's Model for Progressive Brittle Failure

Hahn [2] assumes that this mode of progressive failure happens when the matrix and the interface are so weak that after its failure the "ply" carries no more load. Since the load carrying capacity of the failed ply suddenly reduces to zero, the

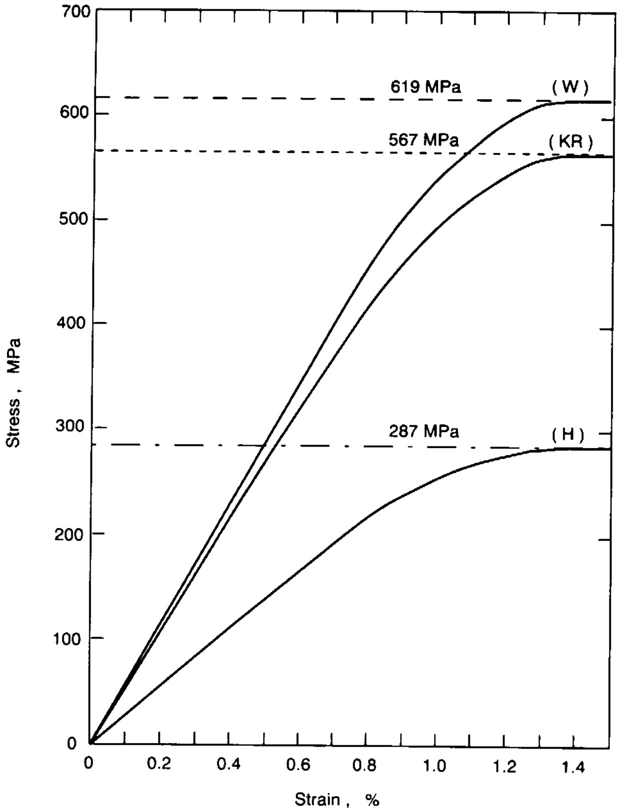

FIG. 5—*Progressive ductile failure of fiber composites with various orientation density functions* (X_t = *1750 MPa,* Y_t = *70 MPa, S = 100 MPa,* E_x = *140 GPa,* E_y = *10 GPa,* G_{xy} = *4 GPa,* ν_{xy} = *0.31)*: (a) *uniform,* H; (b) *nonuniform,* KR; (c) *biased nonuniform,* W.

remaining plies have to readjust their strains in order to take on a redistributed load. Under this assumption the secant modulus at strain ϵ is

$$E_s(\epsilon) = E_o - \frac{2}{\pi}\int_o^{\pi/2} E(\theta)H(\epsilon - \epsilon_o(\theta))d\theta \qquad (32)$$

and hence the stress $\sigma(\epsilon)$ is simply

$$\sigma(\epsilon) = E_s(\epsilon)\cdot\epsilon = E_o\epsilon - \epsilon\frac{2}{\pi}\int_o^{\pi/2} E(\theta)H(\epsilon - \epsilon_o(\theta))d\theta \qquad (33)$$

Since the negative term in the above expression (Eq 33) increases with increasing load, it is conceivable that there will be a value of ϵ beyond which the $\sigma(\epsilon)$ curve no longer increases. Hahn calls this value ϵ_M and then develops a method for finding the value θ_M of the ply in which this happens. For uniformly random composites Hahn's method [2] consists of intersecting two curves, $\bar{E}(\theta)$ and $\bar{X}(\theta)$, where

$$\bar{E}(\theta) = \int_o^{\pi/2} E(\theta)d\theta, \quad \text{and} \quad \bar{X}(\theta) = \frac{X(\theta)}{\epsilon_o(0) - \epsilon_o(\pi/2)} \qquad (34)$$

However, this theory is only applicable for monotonically nonincreasing $\epsilon_0(\theta)$. This is not always the case, as shown in the numerical example of Fig. 4.

Another crucial assumption made by Hahn (and also illustrated in his schematic drawing) is that the slope of the function

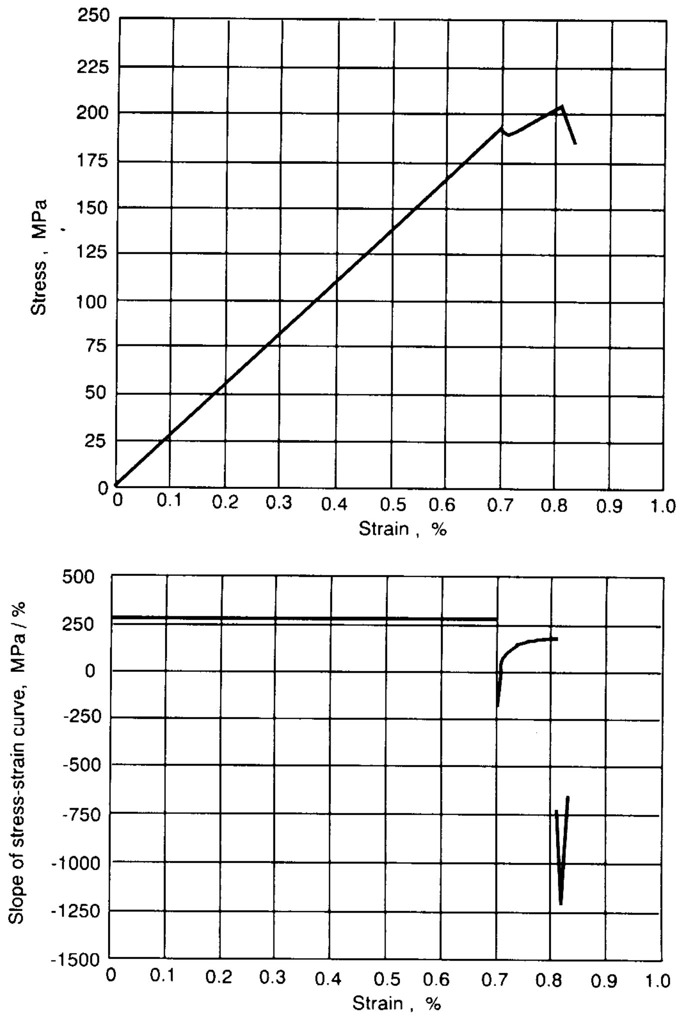

FIG. 6—*Progressive brittle failure of random fiber composite showing slope discontinuity points* (X_t = 1750 MPa, Y_t = 70 MPa, S = 100 MPa, E_x = 140 GPa, E_y = 10 GPa, G_{xy} = 4 GPa, ν_{xy} = 0.31).

$\sigma(\epsilon)$ is continuous. Hence his method was set up from the premise of solving for $d\sigma(\epsilon_M)/d\epsilon = 0$. However, this is not necessarily always possible, since there are values of ϵ for which the derivative $d\sigma(\epsilon)/d\epsilon$ is not defined. To illustrate this, we numerically evaluated Eq 33 over the range $\epsilon \in (0, 1\%)$. Figure 6 shows the resulting $\sigma - \epsilon$ curve; at failure ($\epsilon_{failure} \approx 0.8\%$) the derivative $d\sigma(\epsilon)/d\epsilon$ does not exist, but the curve presents two asymptotic values for the derivative, one just before, and another just after failure has occurred. Physically, this behavior seems possible, since brittle failure can lead to drastic changes in stiffness. Its mathematical aspects will be discussed next.

Extension of Hahn's Model for Progressive Brittle Failure

Let us consider, again, Hahn's general assumption that the failed plies cannot carry loads any longer, and hence a load redistribution occurs after each ply failure. Generalizing for an arbitrary fiber orientation density $h(\theta)$ defined over the range (θ_{min}, θ_{max}), we write the previous equation (32) as

$$E_s(\epsilon) = E_o - \int_{\theta_{min}}^{\theta_{max}} h(\theta)E(\theta)H(\epsilon - \epsilon_o(\theta))d\theta \qquad (35)$$

and hence

$$\sigma(\epsilon) = E_o\epsilon - \epsilon \int_{\theta_{min}}^{\theta_{max}} h(\theta)E(\theta)H(\epsilon - \epsilon_o(\theta))d\theta \quad (36)$$

To find the value at which the $\sigma - \epsilon$ curve has a maximum, we can formally differentiate with respect to ϵ, giving proper care to the special differentiation properties of the generalized functions. Hence we recall that the derivative of Heaviside's function $H(t)$ is Dirac's function $\delta(t)$, with t as a generic variable. Hence one formally writes

$$\frac{d\sigma(\epsilon)}{d\epsilon} = E_o - \int_{\theta_{min}}^{\theta_{max}} h(\theta)E(\theta)H(\epsilon - \epsilon_o(\theta))d\theta - \epsilon \int_{\theta_{min}}^{\theta_{max}} h(\theta)E(\theta)\delta(\epsilon - \epsilon_o(\theta))d\theta \quad (37)$$

The first two terms in the above expression can be identified with the secant modulus $E_s(\epsilon)$ as given by Eq 35, whereas the last term requires special attention due to the presence of Dirac's function under the integral sign. This term will take nonzero values at all θ points at which the current value of ϵ intersects the failure-strain curve $\epsilon_0(\theta)$.

In the restrictive case of a monotonically nonincreasing $\epsilon_0(\theta)$ function there is only one such intersection, as originally assumed by Hahn [2]. This may be the case if ultimate failure takes place at low strain, maintaining the failure point on the monotonically nonincreasing branch of the $\epsilon_0(\theta)$. Hence one would write Eqs 35 and 36 as

$$E_s(\epsilon) = \int_{\theta_{min}}^{\theta_0(\epsilon)} h(\theta)E(\theta)d\theta, \qquad \sigma(\epsilon) = \epsilon \int_{\theta_{min}}^{\theta_0(\epsilon)} h(\theta)E(\theta)d\theta \quad (38)$$

where $\theta_0(\epsilon)$ is the value of θ at which the current strain level intersects the failure strain curve $\epsilon_0(\theta)$. Performing differentiation with respect to ϵ and using the calculus rule for differentiating an integral with limits depending on the differentiation variable, one writes

$$\frac{d\sigma(\epsilon)}{d\epsilon} = \int_{\theta_{min}}^{\theta_0(\epsilon)} h(\theta)E(\theta)d\theta - \epsilon \left[h(\theta_o(\epsilon))E(\theta_o(\epsilon)) \frac{-d\theta_o(\epsilon)}{d\epsilon} \right] \quad (39)$$

The last term of the above expression is due to the variable limits of integration. Using Eq 38 for the definition of $E_s(\epsilon)$ and the usual relation for a 1:1 x to y mapping

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} \quad (40)$$

we write the equation defining the failure strain ϵ_M for a monotonically nonincreasing failure curve as

$$E_s(\epsilon_M) = \frac{\epsilon_M h(\theta_o(\epsilon_M))E(\theta_o(\epsilon_M))}{\left.\dfrac{d\epsilon_o}{d\theta}\right|_{\epsilon_0=\epsilon_M}} \quad (41)$$

Note that the denominator involves the slope of the failure curve at the point of intersection with the applied strain. Hence *the above equation may become singular* if the failure curve is not monotonically decreasing, e.g., if it has a point of local extremum at which the derivative $d\epsilon_0/d\theta$ vanishes. But we have already shown that *"real-life"* $\epsilon_0(\theta)$ curves are not monotonically decreasing *(and not even monotonically nonincreasing)*, and hence Hahn's simplifying assumption does not always hold. In fact, extremum points, and also multiple intersections between the applied strain and the failure curve, are a common occurrence.

Usually, more than one δ spike will appear under the integral sign as the strain range is swept by the applied strain values. The encounter between the current strain level and the "bottoming valley" in the $\epsilon_0(\theta)$ curve will produce a discontinuous variation ("jump") in the value and sign of $d\sigma/d\epsilon$. Such a situation is illustrated in Fig. 6 showing that the ultimate failure is accompanied by a sign change in $d\sigma/d\epsilon$ without necessarily encountering a conventional maximum of the $\sigma(\epsilon)$ curve.

Hence we advocate the use of the more general expression (Eq 36) in all cases under consideration. The mathematical details discussed above are easily handled by the current PC codes, and no particular concern is required by the user. However, one should not seek a closed-form expression for the ultimate strength in this type of failure, since no such expression is possible as long as a singularity occurs in the behavior of $d\sigma(\epsilon)/d\epsilon$. Instead, a plot of the $\sigma - \epsilon$ curve, and of its derivative $d\sigma(\epsilon)/d\epsilon$, would easily identify the ultimate condition, and the corresponding ultimate strength (Fig. 6).

The Influence of Fiber Orientation on the Progressive Brittle Failure Strength

The stress-strain curve for the progressive brittle failure of a uniformly random composite is given in Fig. 7, Curve H. It can be seen that the ultimate strength in this case is

$$\overline{X}_H = 218 \text{ MPa} \quad (42)$$

However, the use of Hahn's original formulation (Eq 34) would have given an ultimate strength of 67.13 ksi (462.79 MPa), which is erroneous. The discrepancy lies in the fact that the $\epsilon_0(\theta)$ curve is not monotonically nonincreasing, and hence the *ultimate strain is only 0.8%, and not 1.34% as predicted by Hahn's theory.* Upon using the *KR* orientation density describing a nonuniformly random composite biased towards the 0° orientation one gets the *KR* stress-strain curve of Fig. 7, and the ultimate strength

$$\overline{X}_{KR} = 421 \text{ MPa} \quad (43)$$

Finally, the strongly biased orientation density *W* having 2% of the fibers purposefully oriented in the 0° direction, yields the *W* stress-strain law of Fig. 7, and the ultimate strength value

$$\overline{X}_W = 461 \text{ MPa} \quad (44)$$

It is worth mentioning that all three stress-strain curves have the same ultimate strain of 0.8%; hence the increased strength is due entirely to the increased modulus through biased fiber orientation. The above three cases show that considerable strength gain (110%) can be obtained in this type of failure using biased dis-

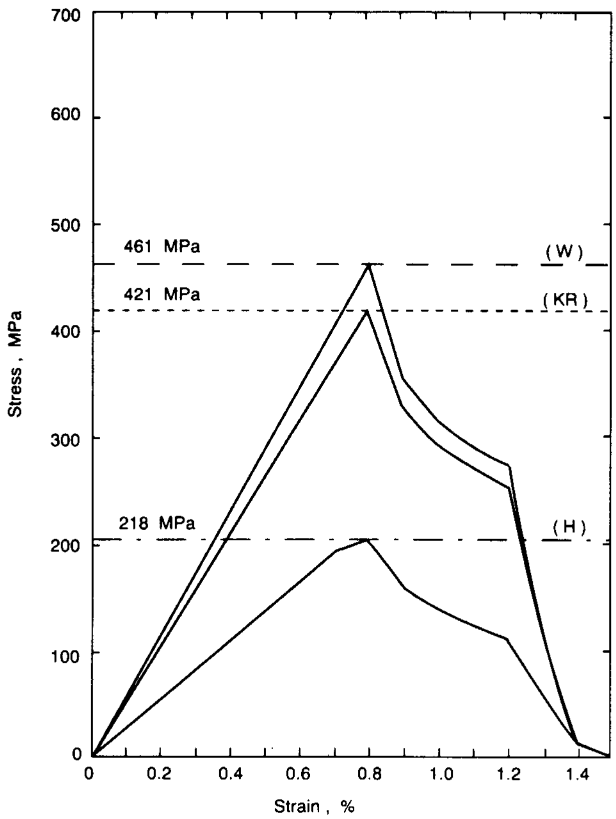

FIG. 7—*Progressive brittle failure of fiber composites with various fiber orientation density functions* (X_t = *1750 MPa*, Y_t = *70 MPa*, S = *100 MPa*, E_x = *140 GPa*, E_y = *10 GPa*, G_{xy} = *4 GPa*, ν_{xy} = *0.31*): (a) *uniform*, H; (b) *nonuniform*, KR; (c) *biased nonuniform*, W.

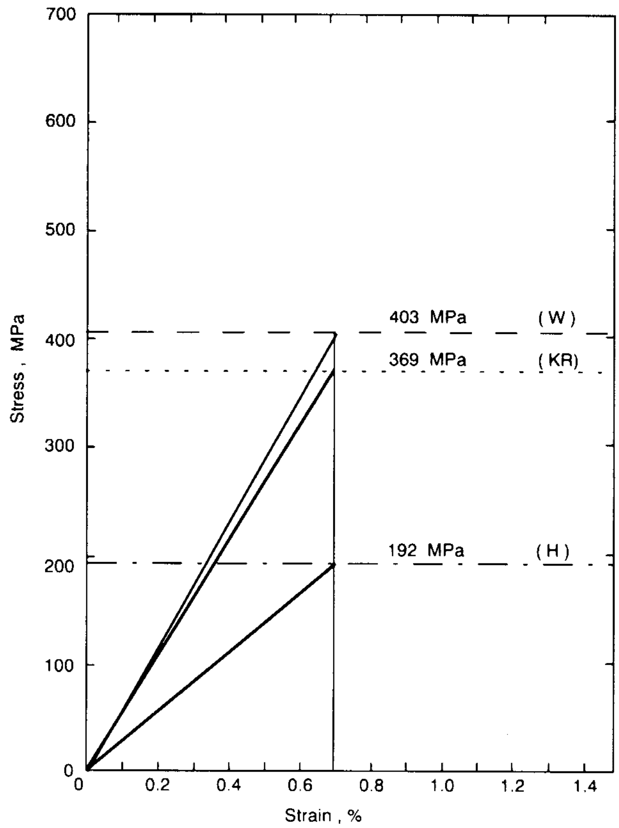

FIG. 8—*Sudden brittle failure of fiber composites with various orientation density functions* (X_t = 1750 MPa, Y_t = 70 MPa, S = 100 MPa, E_x = 140 GPa, E_y = 10 GPa, G_{xy} = 4 GPa, ν_{xy} = 0.31): (a) *uniform*, H; (b) *nonuniform*, KR; (c) *biased nonuniform*, W.

tributions. The differential increase between the two nonuniform distributions under consideration (*KR* and *W*) is small but significant (9%).

Sudden Brittle Failure

Review of Hahn's Model for Sudden Brittle Failure

Hahn [2] assumes that the cracks produced by the failure of the weakest plies propagate across the laminate thickness, leading to the ultimate failure of the complete laminate. In this case the strength of the laminate is given by

$$\bar{X} = E_o\epsilon_o(\theta_L) = \epsilon_o(\theta_L)\,\frac{2}{\pi}\int_o^{\pi/2} E(\theta)d\theta \tag{45}$$

Obviously this formula only applies to the restrictive case of a uniform random composite with θ-symmetry. The special angle θ_L corresponds to the lowest value of $\epsilon_0(\theta)$ and, as Hahn observes [2], this usually coincides with $\theta = \pi/2$.

Extension of Hahn's Model for Sudden Brittle Failure

We extend Hahn's model for sudden brittle failure by letting θ span the arbitrary range (θ_{min}, θ_{max}), and by relaxing the uniform orientation density requirement imposed on $h(\theta)$. The special angle θ_L corresponding to the lowest value of $\epsilon_0(\theta)$ can be found in the general case using the *min* function applied over a range of parameters containing the values of $\epsilon_0(\theta)$ at the two interval ends, $\epsilon_0(\theta_{min})$, $\epsilon_0(\theta_{max})$, and at points of local minimum inside this interval. For example, the numerical case illustrated in Fig. 4 shows a function $\epsilon_0(\theta)$ which has a local minimum at θ_3 (corresponding to the minimum of the shear failure mode), and hence

$$\epsilon_{OL} = \min\,(\epsilon_0(\theta_{min}),\ \epsilon_0(\theta_{max}),\ \epsilon_0(\theta_3)) \tag{46}$$

The use of the exact formula (Eq 46) for predicting ϵ_{OL} is very important when a pronounced low modulus is encountered in the transverse direction and thus the corresponding failure strain at $\theta = \pi/2$ is not necessarily small.

With the above definitions, the stress-strain law for the case of sudden brittle failure takes general form

$$\sigma(\epsilon) = E_o \in H(\epsilon_{OL} - \epsilon),\ E_o = \int_{\theta_{min}}^{\theta_{max}} h(\theta)E(\theta)d\theta \tag{47}$$

where a linear behavior is exhibited up to the ultimate failure. Then the ultimate strength is given by

$$\bar{X} = E_o\epsilon_{OL} \tag{48}$$

The Influence of Fiber Orientation on the Sudden Brittle Failure Strength

The stress-strain curve for the sudden brittle failure of a uniform random composite is given by Curve H of Fig. 8. It can be seen that the ultimate strength in this case is

$$\overline{X}_H = 192 \text{ MPa} \tag{49}$$

Upon using the *KR* orientation density describing a nonuniform random composite biased towards the 0° orientation one gets *KR* curve of Fig. 8, and the ultimate strength

$$\overline{X}_{KR} = 369 \text{ MPa} \tag{50}$$

Finally, the strongly biased orientation density *W*, having 2% of the fibers purposefully oriented in the 0° direction, yields the *W* stress-strain law of Fig. 8, and the ultimate strength value

$$\overline{X}_W = 403 \text{ MPa} \tag{51}$$

It is worth mentioning that all three stress-strain curves have the same ultimate strain of 0.68%; hence the increased strength is due entirely to the increased modulus.

The above three cases show that considerable strength gain (100%) can be obtained in this type of failure using biased distributions. However, the differential increase between the two nonuniform distributions under consideration (*KR* and *W*) is only marginal (9%).

Comparison with Experimental Data

A limited amount of experimental data is available to date on the mechanical properties of random fiber composites [*4,5,14–16*]. No usable experimental data could be found on random composites with clearly defined nonuniformed (or biased) distributions. Hence, our comparison had to be limited to uniform random fiber composites, typically the SMC products. For example, SMC-R50 is a typical polyester-glass random fiber composite [*16*] with 50% weight (35.4% volume) of chopped E-glass strands, 2 in. (6 cm) long. Its matrix consists of 31.7% weight (51.0% volume) isophthalic polyester, and 18.3% weight (13.6% volume) of calcium carbonate filler and various other process materials (inhibitor, initiator, thickener, and mold release agent). Carefully conducted experiments on SMC-R50 material processed under strict quality control conditions were conducted in the late 1970s, and a comprehensive set of mechanical property data were reported [*16*]. Table 1 shows the relevant mechanical properties data (moduli, strengths, and Poisson's ratio) of SMC-R50. Figure 9 presents a typical stress-strain curve [*16*] for this product. One notices the two-moduli shape of the curve, with a "knee" taking place at roughly $\epsilon_{knee} = 0.25\%$. Significantly, the rest of the σ-ϵ curve is straight to 1.5% strain. Extensive nonlinearity takes place on the last portion of the σ-ϵ curve, i.e, over about one tenth of the total recorded strain.

The progressive ductile failure equation (28) was used to model the experimental σ-ϵ curve of Fig. 9. To account for the "knee" behavior a further extension of Hahn's theory was performed, and the following σ-ϵ relation was used

$$\sigma(\epsilon) = (1 - k_E)\sigma_{knee}(\epsilon) + k_E \int_{\theta_{min}}^{\theta_{max}} h(\theta)E(\theta)\min(\epsilon, \epsilon_o(\theta))d\theta \tag{52}$$

TABLE 1—*Calculated and experimental data for SMC-R50.*

Unidirectional	Random	
Estimated	Calculated	Experimental [*16*]
E_x = 34.5 GPa	E_{calc} = 16.0 GPa	E_{exp} = 15.6 ± 0.2 GPa
E_y = 9.7 GPa	E_{calc} = 16.0 GPa	E_{exp} = 15.6 ± 0.2 GPa
G_{xy} = 4.5 GPa	G_{calc} = 5.5 GPa	G_{exp} = 5.9 ± 0.2 GPa
ν_{xy} = 0.28	ν_{calc} = 0.28	ν_{exp} = 0.31 ± 0.02
X_t = 586.5 MPa	X_{calc} = 157.5 MPa	X_{exp} = 164 ± 3 MPa
Y_t = 85 MPa	X_{calc} = 157.5 MPa	X_{exp} = 164 ± 3 MPa
S = 100 MPa	N/A	N/A
N/A	ϵ_{calc}^{ult} = 2.05%	ϵ_{exp}^{ult} = 1.73 ± 0.04%

where k_E is a modulus efficiency factor, and $\sigma_{knee}(\epsilon)$ is given by:

$$\sigma_{knee}(\epsilon) = E_o[(\epsilon - \epsilon_{knee}) H(\epsilon - \epsilon_{knee}) + \epsilon_{knee}] \tag{53}$$

Recalling that the present theory relies on unidirectional properties (E_x, E_y, ν_{xy}, G_{xy}, X_t, Y_t, and S) being known, we note that the SMC-R5O reference [*16*] did not report any such tests. Hence, an iterative inverse method was used to estimate the required constants within the micromechanical-based bounds consistent with the constitutive materials properties and volume fractions. Table 1 presents the values of these estimated constants. It can be seen that very good agreement was obtained in all the reported properties, well within the experimental scatter.

The σ-ϵ curve predicted by our theory is shown in Fig. 10, along with the experimental results [*16*]. The agreement is excellent for most of the curve, and quite good for the pre-failure last 1/10th portion. In view of similar results reported elsewhere,

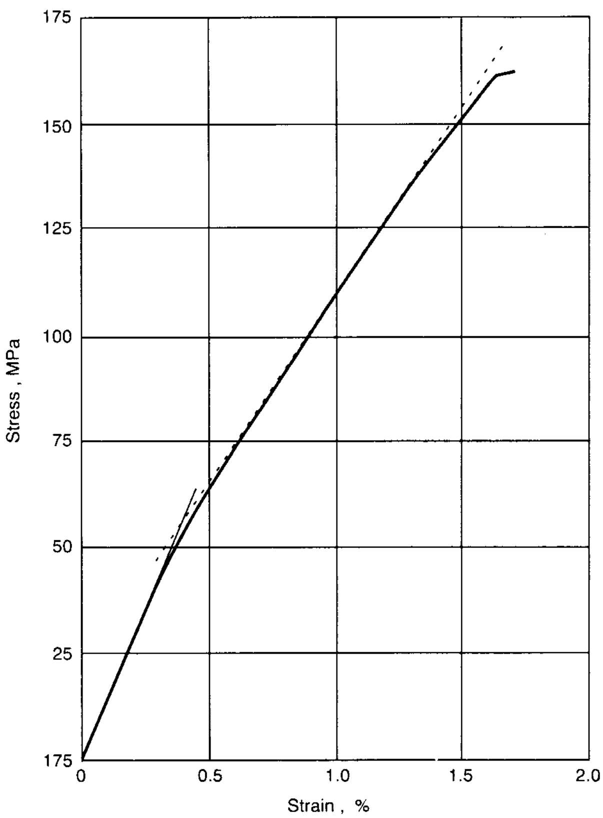

FIG. 9—*Typical stress-strain curve to failure for SMC-R50 at room temperature (23°C) showing the bi-tangent behavior* [16].

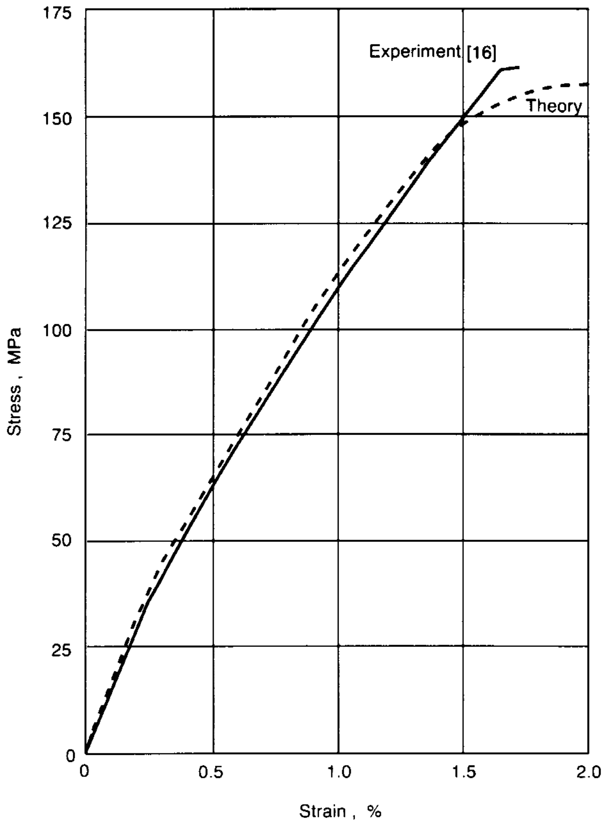

FIG. 10—*Comparison of experimental data* [16] *and theoretical prediction for SMC-R50* (X_t = 586.5 MPa, Y_t = 85 MPa, S = 100 MPa, E_x = 34.5 GPa, E_y = 9.7 GPa, G_{xy} = 4.5 GPa, ν_{xy} = 0.28).

we can say with confidence that the present method has experimental confirmation. However, further experimental investigation should be conducted to determine the exact mechanism of final failure, since other possible phenomena than those considered in the present investigation might be involved (failure of fiber-matrix interface, interphase mechanics, etc.).

Conclusions

Hahn's original work [2] on the strength of random fiber composites has been reviewed and extended. In view of the computational technology available today (on even the simpler PCs), most of the numerical difficulties associated with using proper mathematical expressions were overcome. Hence, more general and comprehensive formulations that properly model the "real life" behavior of this type of composites were derived. The three generally accepted failure modes (progressive ductile, progressive brittle, and sudden brittle) were considered in a mathematically consistent manner, overcoming the limitations imposed by Hahn for computational ease. Thus, a unified theory is now available for further use.

Besides Hahn's uniform orientation density function, other density functions were considered in the numerical examples thus showing the benefits that can be drawn from producing "biased quasi-random composites" (similar to tailoring of the conventional composite designs). More than 100% predicted gains in strength were obtained in the principal load direction.

The maximum stress and maximum strain failure criteria were reassessed in view of their physical insight and possibility for extension. As more failure mechanisms become available through careful and detailed research concerning the fiber-matrix interaction and the interphase, their incorporation in these criteria can be performed with ease. (In contrast, quadratic type criteria like maximum work, Tsai-Hill, Tsai-Wu, etc., do not offer a similar physical insight.) Certain mathematical loop-holes in the use of the maximum strain criterion were also highlighted and means of overcoming them were shown. On balance, the two criteria were shown to coincide on most of the θ range, though our opinion is that the maximum strain criterion is more physically correct, though more complicated in use.

Comparison with experimental data was performed using work reported elsewhere on SMC-R50 uniform random fiber composite. Very good agreement was obtained, confirming the soundness of the present theory. No data were available for comparison in the case of nonuniform and biased random fiber distributions. Obviously, this subject needs to be addressed next, in cooperation with the interested industries.

The work presented so far was limited to unidirectional loading of a laminated random fiber composite. Its purpose was mainly to clear the way and to lay the mathematical foundation for further work. To this end bi- and tri-dimensional loading situations are a natural continuation. More complicated random composites with out-of-plane fiber orientations, and no clearly definable "plies," constitute another worthwhile challenge.

Acknowledgments

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APPENDIX

The Maximum Strain Failure Criterion

Assume an off-axis ply of orientation θ, loaded along the σ_1 direction. Resolving the stresses about the internal principal directions *x, y, s,* we write

$$\sigma_x = \sigma_1 \cos^2\theta, \quad \sigma_y = \sigma_1 \sin^2\theta, \quad \sigma_s = \sigma_1 \cos\theta \sin\theta \qquad (54)$$

Hence the corresponding strains are

$$\epsilon_x = \frac{\sigma_1}{E_x}[\cos^2\theta - \nu_x \sin^2\theta], \quad \epsilon_y = \frac{\sigma_1}{E_y}[\sin^2\theta - \nu_y \cos^2\theta],$$

$$\epsilon_s = \frac{\sigma_1}{G}\cos\theta \sin\theta \qquad (55)$$

For the on-axis ply we assumed the basic strengths to be X_t, X_c, Y_t, Y_c, and S, where subscripts t and c represent tension and compression, respectively. The corresponding failure strains are

$$\epsilon_{xX} = \frac{X}{E_x}, \quad \epsilon_{xY} = -\nu_x\frac{Y}{E_x}, \quad \epsilon_{yX} = -\nu_y\frac{X}{E_y},$$

$$\epsilon_{yY} = \frac{Y}{E_y}, \quad \epsilon_{sS} = \frac{S}{G} \qquad (56)$$

where the subscripts t and c will have to be applied for tension and compression. It is usually assumed that the Poisson effects are less critical than the main loading, and hence the failure strains will be ϵ_{XX}, ϵ_{YY}, and ϵ_{SS}. Thus, for nonfailure the following

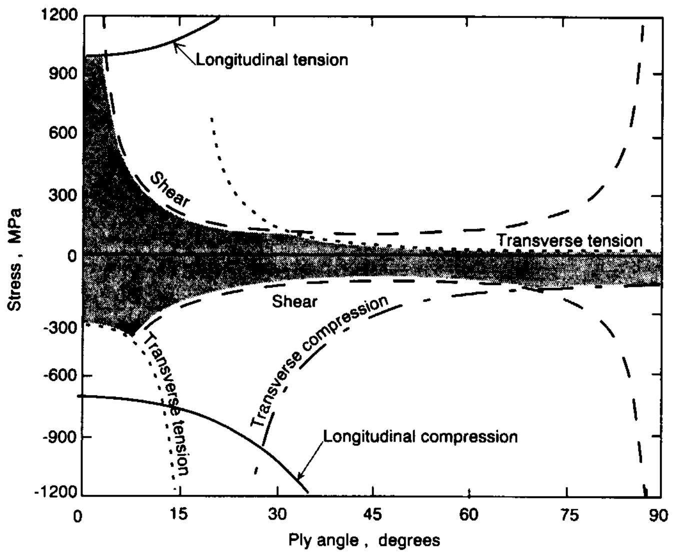

FIG. 11—*Failure stress function for maximum strain criterion showing compression failure through matrix tension (glass-epoxy:* X_t = 1000 MPa, X_c = 700 MPa, Y_t = 28 MPa, Y_c = 140 MPa, S = 55 MPa, E_x = 54 GPa, E_y = 18 GPa, G_{xy} = 8.6 GPa, ν_{xy} = 0.25).

simultaneous inequalities apply.

$$-\frac{X_c}{E_x} < \frac{\sigma_1}{E_x}(\cos^2\theta - \nu_x \sin^2\theta) < \frac{X_t}{E_x},$$

$$-\frac{Y_c}{E_y} < \frac{\sigma_1}{E_y}(\sin^2\theta - \nu_y \cos^2\theta) < \frac{Y_c}{E_y} - \frac{S}{G} < \frac{\sigma_1}{G} < \frac{S}{G} \quad (57)$$

The above expressions have two critical values θ_x and θ_y given by

$$\theta_x = \tan^{-1}\sqrt{1/\nu_x} \qquad \theta_y = \tan^{-1}\sqrt{\nu_y} \quad (58)$$

and hence the sign changes must be properly accounted for when the inequalities are resolved. If the above ordering assumptions for the on-axis failure strains apply (Poisson effects less than main loading) then the failure function for σ_1 can be defined as

$$X(\theta) = \min\left(\frac{X}{\cos^2\theta - \nu_x \sin^2\theta}, \frac{Y}{\sin^2\theta - \nu_y \cos^2\theta}, \frac{S}{\cos\theta \sin\theta}\right) \quad (59)$$

where subscripts t and c apply for tension and compression loading, respectively. The positive branches of the X and Y dependent functions are assumed.

However, there are cases when some of the assumptions used in ordering the inequalities are not satisfied. For example, the matrix tension strength Y_t and the transverse Poisson's ratio ν_y may be such that under overall compression loading the Poisson effects induce matrix failure in tension before fiber failure in compression occurs ($Y_t/\nu_y < X_c$). Then the compression failure will have the mixed formulation

$$X_c(\theta) = \min\left(-\frac{Y_t}{\sin^2\theta - \nu_y \cos^2\theta}, \frac{Y_c}{\sin^2\theta - \nu_y \cos^2\theta}, \frac{S}{\cos\theta \sin\theta}\right) \quad (60)$$

where the negative branch of Y_t function is taken below θ_y, and the positive branch of Y_c above θ_y. A numerical illustration of this case is given in Fig. 11. A detailed discussion of the maximum strain criterion special features was given by Wu [*17*], and some interesting considerations were presented by Wetherhold and Woonbong [*9*].

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