LAMB WAVE GENERATION AND DETECTION WITH PIEZOELECTRIC WAFER ACTIVE SENSORS

by

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ABSTRACT

Structural Health Monitoring (SHM) of critical structural parts plays a very important role for preventing structural failure and loss of human lives. Current defect detection methods require bulky, expensive transducers and equipments. These limitations make them not suitable for in-situ, online SHM. In response to this need, the use of piezoelectric wafer active sensors (PWAS) and Lamb wave propagation for structural health monitoring has been proposed.

The goal of the research was to develop the scientific and engineering basis for the use of PWAS and the Lamb wave propagation method in structural health monitoring.

The fundamental theory of Lamb wave phenomenon was reviewed. The derivation of the characteristic equations and wave speed dispersion curve of Lamb waves are provided. The complex behavior of the Lamb wave modes at different frequency ranges was explained. PWAS were studied from both theoretical and practical aspects. Based on these theories, the interaction of thin plates and PWAS was studied.

In the experimental study, PWAS were bonded on thin aluminum alloy plates to evaluate its capability in Lamb wave generation and detection. The effects of excitation signal modulation and signal processing techniques are discussed. Signal propagation speed was calculated from the experimental data, which agree with the theoretical prediction. Operations similar to those used in conventional ultrasonic testing, e.g. pitch-catch and pulse-echo methods, were tested.
A PWAS phased array was proposed to perform 2-D scanning of large areas. The algorithm for generating a virtual scanning beam and offline signal focusing were derived. A prototyping PWAS array was constructed on aluminum alloy plates with simulated fatigue cracks. With the proposed phased array algorithm, cracks were detected successfully.
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CHAPTER 1 INTRODUCTION

This Dissertation describes the application of piezoelectric wafer active sensors (PWAS) in generating and detecting Lamb waves. The research work was concentrated on the development of the theoretical background of the interaction between the piezoelectric wafer active sensors and Lamb waves, and then addresses issues of its practical implementation. Examples of piezoelectric wafer active sensor generating Lamb wave in thin-wall metallic structures and detecting structural defects are presented.

1.1 MOTIVATION

Health monitoring of aging structures is a major concern of the engineering community. This need is even more intense in the case of aging aerospace structures which have been operating well beyond their initial design life. Multi-site fatigue damage, hidden cracks in hard-to-reach locations and corrosion are among the major flaws encountered in today’s extensive fleet of aging aircraft and space vehicles. The durability and health monitoring of such structures form the subject of extensive research in many universities, government labs, and industry. This area is of growing concern and worthy of new and innovative approaches. The nation’s safety and reliability record is excellent but the fatigue of its aging aerospace fleet is raising major concerns. Though well established design and maintenance procedures exist to detect structural fatigue, new and unexpected phenomena must be accommodated by the application of advanced flaw detection methods.
One example is the case of the Aloha Airlines 1988 accident (Figure 1.1). This accident was due to a relatively new phenomenon, multi-site crack damage in the skin panel joints, resulting in catastrophic “un-zipping” of large fuselage panels.

![Fuselage panels were ripped away](image)

**Figure 1.1** Aloha Airlines Boeing 737 accident on April 28, 1988 was due to multi-site crack damage in the skin panel joints resulting in catastrophic “un-zipping” of large portions of the fuselage. ([http://the.honoluluadvertiser.com/2001/Jan/18/118localnews1.html](http://the.honoluluadvertiser.com/2001/Jan/18/118localnews1.html))

Another example is a recent deadly accident of C-130A fire fighter crashed in Walker, California (Figure 1.2). During the retardant drop the airplane’s wings folded upwards at the center wing-to-fuselage attachment point and separated. The right wing folded just before the left wing. The three flight crewmembers were killed and the airplane was destroyed. During the investigation fatigue cracks were found in pieces of the wing structure in the C-130A. The preliminary results indicate that widespread fatigue was not evident over the entire wing. On September 26, 2002 the Federal Aviation Administration ordered wing inspections on all C-130A air tankers after NTSB investigators determined that cracks in the wings caused the C-130A crash. The Safety Board and the FAA are evaluating current crack detection techniques to determine their reliability.
These accidents compelled the aerospace engineering community to take a fresh look at the fail-safe, safe-life, and damage tolerance design philosophies. The effect of aging on aircraft airworthiness and the deadly combination of fatigue and corrosion had to be reassessed. Prevention of such unexpected occurrences could be improved if on-board health monitoring systems exist that could assess the structural integrity and would be able to detect incipient damage before catastrophic failures occur. To gain wide spread acceptance, such a system has to be cost effective, reliable, compact and light weight.
Another important aspect related to the operation and maintenance of our aging aircraft fleet is that of cost. One of the main reasons for this rise in cost is that most of the inspections and structural health monitoring is performed manually. As aircrafts age additional tasks such as supplemental structural inspections are required. These increase the costs of maintaining an aging fleet. Select use of condition-based maintenance coupled with sensitive and continuous on-line monitoring of structural integrity would significantly reduce the cost of inspection programs. “Retirement for cause” instead of “retirement as planned” could reduce the cost while maintaining a safe operation life for many aging aircraft structures. The replacement of our present-day manual inspection with automatic health monitoring would substantially reduce the associated life-cycle costs. Hence, there is a need for reliable structural health monitoring systems that can automatically process data, assess structural condition, and signal the need for human intervention.

1.2 RESEARCH GOAL, SCOPE AND OBJECTIVES

The goal of the research is to develop a method to use piezoelectric-wafer active sensors (PWAS) and Lamb wave for thin-wall structural health monitoring, and damage detection. The scope of this dissertation is to address the issues of Lamb wave modeling, the theoretical analysis of sensor-structure interaction for 1-D and 2-D geometries, the investigation of suitable damage identification algorithms for data processing, and the Lamb wave propagation structural health monitoring methodology. The objectives for this research are defined as follows:
1. To present the detailed modeling of Lamb wave and its properties at various frequencies and modes. Validate the modeling through experimental testing, and address the issues of Lamb wave generation and detection.

2. To develop analytical models and perform validation experiments for the sensor-structure interaction revealing the Lamb wave generating and sensing mechanism of PWAS.

3. To present an algorithm that enables an array of PWAS to interrogate a large area of structure.

4. To demonstrate the applications of the PWAS and Lamb wave propagation method for Structural Health Monitoring (SHM).
PART I GUIDED WAVES IN THIN WALL STRUCTURES
CHAPTER 2 STATE OF THE ART IN GUIDED WAVE RESEARCH

Guided wave is the name of a family of waves that require boundary(s) for their existence. The members of this family are named after the investigators who did great contributions to the understanding of these waves. A few examples are the Rayleigh waves, Love waves, Lamb waves, and Stonley waves. Rayleigh wave is the wave that propagates on the traction-free surface of a semi-infinite solid (Rayleigh, 1885). Love wave exist in a half-space covered by a layer of material with different elastic properties (Love, 1926). Lamb waves are the waves that propagate in plates with traction-free surfaces (Lamb, 1917). Stonley waves are free waves that occur at an interface between two media (Stonley, 1924). Some of these waves have analytical solutions, and were studied thoroughly more than a century ago, e.g., the Rayleigh wave. Some can not be solved analytically, e.g., the Lamb waves. At the time of it was first investigated, only a few data points were given on the Lamb wave dispersion curves. With the help of some special solutions of characteristic equations, i.e., the Rayleigh-Lamb equations, Mindlin (1960) plotted the frequency-wave number relation graph of Lamb wave modes approximately. With the development of computers, numerical solutions were obtained. Viktorov (1967) presented the Lamb wave phase velocity and group velocity dispersion curves, which were calculated by a computer. A general procedure of calculating the Lamb wave speed dispersion curve was provided by Rose (1999), which can be realized in numerical programming packages, e.g. Matlab, and run on personal computers.
Guided waves have the capability of propagating along the wave guide for very long distances. This makes them good choices for many applications. Because thin-wall structures are very commonly used, Lamb waves are of particular interests in structural health monitoring. Derivation of the Rayleigh-Lamb equation and analysis of the Lamb wave behaviors can be found in many text books (Viktorov, 1967; Auld, 1973; Graff, 1975; Achenbach, 1984; Rose, 1999).

Auld (1973) studied the orthogonality of Lamb wave modes and presented an algorithm for calculating the amplitude of each mode. Rose (1999) studied the structure of Lamb waves and explained the complex displacement pattern of Lamb wave modes. The result of these studies are used in choosing Lamb wave mode for different applications and selective excitation of these modes.

Finite Element Method (FEM) and Boundary Element Method (BEM) are used by many researchers to study the behavior of guided wave in structures of various shapes and composites. Cho and Rose (2000) studied guided wave interactions with a surface braking defect. Hybrid boundary element method was used in combination with an elastodynamic boundary integral equations and Lamb wave normal mode expansion. Zhao and Rose (2003) presented wave scattering analysis implemented by boundary element methods and the normal mode expansion technique in studying the sizing potential of two-dimensional shaped defects in a wave guide. Seifried et al.,(2002) used analytical and computational models to develop a quantitative understanding of the propagation of guided Lamb waves in multi-layered, adhesive bonded components. FEM was used to numerically model the transient Lamb wave propagation problem.
CHAPTER 3 SH WAVE IN PLATES

Before considering Lamb waves in plates, shear horizontal (SH) waves are studied, because they are much simpler than the Lamb waves. In SH waves, particle motion is perpendicular to wave propagating direction, and parallel to the surface of the medium.

Consider a plate of thickness $2d$ and infinite extent in width and length (Figure 3.1).

![Figure 3.1 SH wave in plate of thickness 2d](image)

Select the coordinate system such that the $xz$ plane is at the middle surface of the plate.

The particle displacement exists only in $z$ direction, while the wave propagates in the $x$ direction. The governing equation is

$$
\nabla^2 u_z(x, y, t) = \frac{1}{c_s^2} \frac{\partial^2 u_z}{\partial t^2}
$$

(3.1)

where the Laplacian operator is defined by

$$
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
$$

(3.2)
The $u_z$ displacement is nonzero ($u_z \neq 0$), while the other two displacements are zero ($u_x = u_y = 0$). The plane strain condition $\partial / \partial z = 0$ applies. The boundary condition is

$$\sigma_{yz} \bigg|_{y=z} = 0$$  \hspace{1cm} (3.3)

The wave is harmonic, so the solution of the wave equation is given by:

$$u_z(x, y, t) = h(y)e^{i(\xi x - \omega t)}$$  \hspace{1cm} (3.4)

where $h(y)$ describes the displacement distribution in the thickness of the plate.

Substitute Equation (3.4) into (3.1). For simplicity, omit the $e^{i(\xi x - \omega t)}$ terms. Hence:

$$\frac{d^2}{dy^2} h(y) + h(y)(i\xi)^2 = \frac{1}{c_s^2} (-i\omega)^2 h(y)$$  \hspace{1cm} (3.5)

Let

$$\beta^2 = \frac{\omega}{c_s^2} - \xi^2$$  \hspace{1cm} (3.6)

Equation (3.5) is simplified to

$$\frac{d^2}{dy^2} h(y) + \beta^2 h(y) = 0$$  \hspace{1cm} (3.7)

This is a homogeneous ordinary differential equation (ODE), which has the solution

$$h(y) = A_1 \sin \beta y + A_2 \cos \beta y$$  \hspace{1cm} (3.8)

Hence, the displacement $u_z$ is

$$u_z = (A_1 \sin \beta y + A_2 \cos \beta y)e^{i(\xi x - \omega t)}$$  \hspace{1cm} (3.9)
The two arbitrary constants, $A_1$ and $A_2$, are determined by the boundary conditions at the upper and lower surfaces. Recall the boundary condition in Equation (3.3),

$$\sigma_{yz} \bigg|_{y=d} = 0$$

where

$$\sigma_{yz} = \mu \varepsilon_{yz}$$

The boundary condition requires that

$$\varepsilon_{yz} \bigg|_{z=d} = \frac{\partial}{\partial y} u_x \bigg|_{y=d} = 0$$  \hspace{1cm} (3.10)

Substituting Equation (3.9) into Equation (3.10) yields

$$\begin{cases} 
A_1 \cos \beta d - A_2 \sin \beta d = 0 & \text{at } y = d \\
A_1 \cos(-\beta d) - A_2 \sin(-\beta d) = 0 & \text{at } y = -d 
\end{cases}$$  \hspace{1cm} (3.11)

To solve for $A_1$ and $A_2$, the determinant of the coefficients matrix of the equations must be zero. Hence

$$\begin{vmatrix} 
\cos \beta d & -\sin \beta d \\
\cos(-\beta d) & -\sin(-\beta d) 
\end{vmatrix} = 2 \cos \beta d \sin \beta d = 0$$  \hspace{1cm} (3.12)

Equation (3.12) is satisfied by

$$\sin \beta d = 0 \rightarrow \beta d = 2n \frac{\pi}{2}, \ n = 0,1,2,... \ (\text{symmetric mode})$$  \hspace{1cm} (3.13)

or

$$\cos \beta d = 0 \rightarrow \beta d = (2n+1) \frac{\pi}{2}, \ n = 0,1,2,... \ (\text{antisymmetric mode})$$  \hspace{1cm} (3.14)
The symmetric and antisymmetric modes refer to the displacement in Equation (3.9). It is either antisymmetric or symmetric with respect to the middle surface \((y = 0)\).

In Equation (3.13), when \(\sin \beta d = 0, \cos \beta d \neq 0\). From Equation (3.11), \(A_1\) must go to zero. Hence, the displacement is given by

\[ u_z = A_2 e^{i(\xi z - \omega t)} \cos \beta y \]  

(3.15)

which is the symmetric mode.

Similarly, when \(\cos \beta d = 0, \sin \beta d \neq 0\). From Equation (3.11), \(A_2\) must go to zero.

Hence, the displacement is given by

\[ u_z = A_2 e^{i(\xi z - \omega t)} \sin \beta y \]  

(3.16)

which is the antisymmetric mode.

From Equations (3.9), (3.13), and (3.14), we can see that the propagating SH wave only exists for certain values of the \(\beta d\) product. Because \(\beta\) is a function of the circular frequency \(\omega\), SH waves only exist at certain frequencies for a plate with a given thickness.

Since

\[ \beta^2 = \frac{\omega^2}{c_s^2} - \xi^2 \]  

(3.17)

\[ \xi = \frac{\omega}{c} \]

then
\[(\beta d)^2 = \left(\frac{\omega d}{c_s}\right)^2 - \left(\frac{\omega d}{c}\right)^2 = (2\pi)^2 \left(\frac{f d}{c_s}\right)^2 - \left(\frac{fd}{c}\right)^2 \]  \hspace{1cm} (3.18)

From Equations (3.13), (3.14) and (3.18), the wave speed can be calculated.

For symmetric modes, Equation (3.13)

\[\beta d = 2n \frac{\pi}{2} \rightarrow (\beta d)^2 = (2\pi)^2 \left(\frac{n}{2}\right)^2 \]

Hence,

\[\left(\frac{fd}{c_s}\right)^2 - \left(\frac{fd}{c}\right)^2 = \left(\frac{n}{2}\right)^2 \]

Therefore, the wave speed for symmetric SH wave is

\[\frac{c}{c_s} = \left[1 - \left(\frac{n}{2}\right)^2 \left(\frac{fd}{c_s}\right)^2\right]^{\frac{1}{2}} \]  \hspace{1cm} (3.19)

Similarly, for the antisymmetric modes, Equation (3.14)

\[\beta d = (2n + 1) \frac{\pi}{2} \rightarrow (\beta d)^2 = (2\pi)^2 \left(\frac{2n+1}{4}\right)^2 \]

Hence,

\[\left(\frac{fd}{c_s}\right)^2 - \left(\frac{fd}{c}\right)^2 = \left(\frac{2n+1}{4}\right)^2 \]

Therefore, the wave speed for antisymmetric SH wave is
\[
\frac{c}{c_s} = \left[1 - \left(\frac{2n+1}{4}\right)^2 \left(\frac{fd}{c_s}\right)^2\right]^{\frac{1}{2}}
\] (3.20)

A plot of the phase velocity of SH wave is shown in Figure 3.2. Non-dimensional values \(c/c_s\) and \(fd/c_s\) are used here, such that it can be used for different materials with different \(c_s\).

![Figure 3.2: Phase velocity dispersion curve of SH wave. Solid lines are for symmetric modes, and dotted lines are for antisymmetric modes.](image)

The group velocity is given by

\[
c_g = \frac{d\omega}{d\xi}
\]

where \(\omega\) is the circular frequency, and \(\xi\) is the wave number. By definition,

\[
\beta^2 = \frac{\omega^2}{c_s^2} - \xi^2 \rightarrow \xi^2 = \frac{\omega^2}{c_s^2} + \beta^2
\] (3.21)
From Equations (3.13) and (3.14), $\beta$ only depends on modes and mode numbers. Taking derivative on both sides of Equation (3.21)

$$2\xi d\xi = 2 \frac{\omega}{c_s^2} d\omega \rightarrow c_g = \frac{d\omega}{d\xi} = \frac{\xi}{\omega} \frac{c_s^2}{\omega} = \frac{c_s^2}{c}$$

Hence,

$$\frac{c_g}{c_s} = \frac{c_s}{c} = \left(\frac{c}{c_s}\right)^{-1} \quad (3.22)$$

The group velocity curves are plotted in Figure 3.3.

![Figure 3.3](image)

**Figure 3.3** Group velocity dispersion curve of SH wave. Solid lines are for symmetric modes, and dotted lines are for antisymmetric modes.

It has been shown in Equations (3.19) and (3.20) that the SH wave velocity is function of frequency and thickness of the plate. From another perspective, the frequency-wave number curve can also reveal the dispersive properties of SH waves.
Since, $\beta^2 = \frac{\omega^2}{c_s^2} - \xi^2$, then

\[
(\beta d)^2 = \frac{(\omega d)^2}{c_s^2} - (\xi d)^2
\]

(3.23)

For the antisymmetric modes,

\[
\beta d = (2n+1)\frac{\pi}{2}
\]

(3.24)

It can be derived that

\[
(2n+1)^2\left(\frac{\pi}{2}\right)^2 = \frac{(\omega d)^2}{c_s^2} - (\xi d)^2
\]

(3.25)

By letting

\[
\Omega^2 = \left(\frac{\omega d}{c_s}\right)^2, \quad \Xi^2 = \left(\frac{\xi d}{\pi}\right)^2
\]

Equation (3.23) is simplified to

\[
(2n+1)^2 + \Xi^2 = \Omega^2
\]

(3.26)

For the symmetric modes,

\[
\beta d = 2n\frac{\pi}{2}
\]

(3.27)

It can be derived that
\[(2n)^2 \left( \frac{\pi}{2} \right)^2 = \frac{(\omega d)^2}{c_s^2} - (\xi d)^2 \quad (3.28)\]

by letting

\[\Omega^2 = \left( \frac{\omega d}{c_s} \right)^2 \left( \frac{\pi}{2} \right)^2\]

\[\bar{\xi}^2 = \left( \frac{\xi d}{\pi} \right)^2 \left( \frac{\pi}{2} \right)^2\]

Equation (3.23) is simplified to

\[(2n)^2 + \bar{\xi}^2 = \Omega^2 \quad (3.29)\]

A plot of this equation is shown in Figure 3.4.

![Figure 3.4](image)

**Figure 3.4** Frequency-wave number curves for SH wave in a plate. Solid lines are for symmetric modes, dashed lines are for antisymmetric modes. (After Graff, 1991)
By observing the frequency-wave number curves (Figure 3.4), the lowest symmetric mode \((n = 0)\) is a straight line which means that the wave number is linearly proportional to the frequency, i.e., \(\frac{\xi}{\omega} = \text{constant}\). Because \(c = \frac{\xi}{\omega}\), so \(c = \text{constant}\), i.e., the phase velocity for the lowest SH symmetric mode does not change with frequency, i.e. it is non-dispersive.

For all other mode numbers \((n = 1,2,3,\ldots)\), the frequency spectrum is a parabolic curve for positive wave numbers. Hence, the wave speed varies with frequency, i.e., they are dispersive.

The wave numbers can take imaginary values, as shown in Figure 3.4. When the wave number takes imaginary values, the displacements in Equation (3.9) becomes

\[
\mathbf{u}_x = (A_1 \sin \beta y + A_2 \cos \beta y) e^{\xi^* x} e^{jnt}
\]  

(3.30)

where \(\xi^* = i\xi\) is a real value. Depending on the sign of \(\xi^*\), the displacement will increase or decrease exponentially with the increase of \(x\). In either case, the displacement will not be periodic in the \(x\) direction. For the former case, \(\xi^* > 0\), and the amplitude of the displacement will go to infinity with the increase of distance. This case does not have physical meaning. For the latter case, \(\xi^* < 0\), the displacements only exist in a small distance. This case is called the "evanescent" mode, which is a non-propagation mode.

From Figure 3.4, it is noticed that higher wave modes appears when the frequency grow over a certain value. These frequencies are called "cut-off" frequencies. At a certain frequency \(\omega\), only those wave modes that have a cut-off frequency lower than \(\omega\) can
exist. Hence, a finite number of propagating SH wave modes can exist for a given frequency.
CHAPTER 4 LAMB WAVES

Lamb waves are guided waves that exist in thin walled structures. Because this type of wave can travel long distance with little attenuation, they have been studied intensively for structural health monitoring, especially in the past few decades.

Early studies of wave propagation in plates were carried out by Rayleigh (1945) and Lamb (1917). The Rayleigh-Lamb theory applies to the propagation of continuous, straight crested waves in infinite plate with free surfaces. Rayleigh-Lamb frequency equations give the relationship between frequency and wave number. The displacement and stress distribution functions can be obtained after the Rayleigh-Lamb frequency equations are solved. Goodman (1952) studied the problem of circular crested waves, and found that the Rayleigh-Lamb frequency equations hold for this case too. Although the displacement and stress distribution of circular crested wave are initially different, these waves converge to the straight crested wave rapidly after a short distance (2~3 wavelength).

Some important properties of Lamb waves need to be determined. First, phase velocity is the fundamental characteristic one needs to obtain. Then, the group velocity can be derived from the relation \( c_g = c - \lambda \frac{\partial c}{\partial \lambda} \), where \( c_g \) is the group velocity, \( c \) is the phase velocity, and \( \lambda \) is the wavelength of the Lamb wave.

When Lamb waves are transmitted, particles move in one of two different ways. If the particle motion is symmetric with respect to the mid surface, it is called symmetric Lamb
waves \((S_0, S_1, S_2\ldots)\). If the particle motion is anti-symmetric with respect to the mid surface, it is called anti-symmetrical Lamb waves \((A_0, A_1, A_2\ldots)\). Furthermore, a number of modes exist for each type of the Lamb waves. Symmetric and anti-symmetric Lamb waves have different phase and group velocities, as well as distribution of particle displacement and stress through the plate thickness.

Lamb waves are dispersive, which means that their phase and group velocities vary with frequency. Phase velocity and group velocity dispersion curves can be drawn as functions of frequency. Because multiple Lamb wave modes can exist at a given frequency, a family of dispersion curves is obtained.

4.1 Symmetric and Antisymmetric Modes

Lamb waves are present in plates with traction-free surfaces. Following Graff (1975), we consider a plate with thickness \(2d\), and a plane strain wave system propagating in the \(x\) direction. (Figure 4.1)

![Figure 4.1](image)

(a) A plate with thickness \(2d\), extends infinitely in \(x\) and \(z\) direction; (b) free body diagram of a small block on the surface of the plate.
The traction-free surface condition can be described as: \( \sigma_{xy} = \sigma_{yx} = \sigma_{yz} = 0 \) at \( y = \pm d \).

The plan strain condition implies \( \frac{\partial}{\partial z} = 0 \) and satisfies identically the condition \( \sigma_{yz} = 0 \).

Assume displacement as function of scalar potential \( \Phi \) and vector potential \( \mathbf{H} \), such that

\[
\mathbf{u} = \nabla \Phi + \nabla \times \mathbf{H}, \quad \nabla \cdot \mathbf{H} = 0
\]  

(4.1)

where \( \nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \), and \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) are the unit vectors in \( x, y, \) and \( z \) directions respectively. The first term in \( \mathbf{u} \) is the gradient of the scalar potential \( \Phi \), while the second term is the curl of the zero-divergence vector potential \( \mathbf{H} \). Expand Equation (4.1) to get:

\[
\mathbf{u} = \frac{\partial \Phi}{\partial x} \mathbf{i} + \frac{\partial \Phi}{\partial y} \mathbf{j} + \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \mathbf{i} + \left( \frac{\partial H_z}{\partial z} - \frac{\partial H_z}{\partial x} \right) \mathbf{j} + \left( \frac{\partial H_x}{\partial x} - \frac{\partial H_x}{\partial y} \right) \mathbf{k}
\]  

(4.2)

For the case in discussion,

\[
\begin{align*}
\mathbf{u}_x &= \frac{\partial \Phi}{\partial x} + \frac{\partial H_z}{\partial y} \\
\mathbf{u}_y &= \frac{\partial \Phi}{\partial y} - \frac{\partial H_z}{\partial x}
\end{align*}
\]  

(4.3)

The wave equations in terms of \( \Phi \) and \( H_z \) are written as

\[
\nabla^2 \Phi = \frac{1}{c_p^2} \frac{\partial^2 \Phi}{\partial t^2}
\]

\[
\nabla^2 H_z = \frac{1}{c_s^2} \frac{\partial^2 H_z}{\partial t^2}
\]  

(4.4)

where \( c_p \) and \( c_s \) are the principle (pressure) and secondary (shear) wave velocities.

Consider the harmonic solution of the wave equations in the form

\[
\]
\[ \Phi = f(y)e^{i(\xi z - \omega t)} \]
\[ H_z = i h_z(y)e^{i(\xi z - \omega t)} \]  

(4.5)

where \( i^2 = -1 \). Substitution of Equation (4.5) into Equation (4.4) gives:

\[ \frac{d^2 f}{dy^2} + \alpha^2 f = 0 \]
\[ \frac{d^2 h_z}{dy^2} + \beta^2 h_z = 0 \]

(4.6)

where \( \alpha^2 = \frac{\omega^2}{c^2} \xi^2 \), and \( \beta^2 = \frac{\omega^2}{c^2} \xi^2 \). Solving (4.6) to get:

\[ f = A \sin \alpha y + B \cos \alpha y \]
\[ h_z = C \sin \beta y + D \cos \beta y \]

(4.7)

Hence,

\[ \Phi = (A \sin \alpha y + B \cos \alpha y)e^{i(\xi z - \omega t)} \]
\[ H_z = i(C \sin \beta y + D \cos \beta y)e^{i(\xi z - \omega t)} \]

(4.8)

Substitution of \( \Phi \) and \( H_z \) from Equation (4.8) into Equations (4.3) yields

\[ u_x = \frac{\partial \Phi}{\partial x} + i \frac{\partial H_z}{\partial y} = i \left[ \xi (A \sin \alpha y + B \cos \alpha y) + \beta (C \cos \beta y - D \sin \beta y) \right] e^{i(\xi z - \omega t)} \]
\[ u_y = \frac{\partial \Phi}{\partial y} - i \frac{\partial H_z}{\partial x} = \left[ \alpha (A \cos \alpha y - B \sin \alpha y) + \xi (C \sin \beta y + D \cos \beta y) \right] e^{i(\xi z - \omega t)} \]

(4.9)

Collecting the sine and cosine terms in equation (4.9) gives

\[ u_x = i(\xi B \cos \alpha y + \beta C \cos \beta y)e^{i\psi} + i(\xi A \sin \alpha y - \beta D \sin \beta y)e^{i\psi} \]
\[ u_y = -(\alpha B \sin \alpha y - \xi C \sin \beta y)e^{i\psi} + (\alpha A \cos \alpha y + \xi D \cos \beta y)e^{i\psi} \]

(4.10)

where \( \psi = (\xi x - \omega t) \) is the phase.
The sine terms are anti-symmetric in \( y \), while the cosine terms are symmetric to \( y \).

Consider the particle motion in a plate, the corresponding symmetric and anti-symmetric motions are illustrated in Figure 4.2.

![Symmetrical motion](image)

**Figure 4.2** Symmetric and anti-symmetric particle motion across the plate thickness.

Note that for symmetric motion with respect to the mid plane, \( u_x \) follows to cosine behavior, while \( u_y \) follows the sine behavior. Similarly, for anti-symmetric motion, \( u_x \) follows the sine behavior, while \( u_y \) follows to cosine behavior. Letting \( A = D = 0 \) in Equations (4.10) yields the symmetric motion, whereas \( B = C = 0 \) yields the anti-symmetric motion.

4.2 SYMMETRIC MOTION

Assume displacement symmetric with respect to the plate mid plate:

\[
\begin{align*}
  u_x &= i(\xi B \cos \alpha y + \beta C \cos \beta y)e^{iy} \\
  u_y &= -(\alpha B \sin \alpha y - \xi C \sin \beta y)e^{iy}
\end{align*}
\]  

(4.11)

The corresponding stresses can be found from Navier’s equation (Graff, 1975)
\[ \sigma_{xx} = (\lambda + 2\mu) \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) - 2\mu \frac{\partial u_y}{\partial y} \]

\[ \sigma_{yy} = (\lambda + 2\mu) \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) - 2\mu \frac{\partial u_x}{\partial x} \]  \hspace{1cm} (4.12)

\[ \sigma_{xy} = \mu \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \]

where \( \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \), and \( \mu = \frac{E}{2(1+\nu)} \) are the Lamé constants, \( E \) is the elastic modulus and \( \nu \) is the Poisson's ratio. The derivatives of \( u_x \) and \( u_y \) are calculated as

\[ \frac{\partial u_x}{\partial x} = -(\xi^2 B \cos \alpha y + \xi \beta C \cos \beta y) e^{iy} \]

\[ \frac{\partial u_x}{\partial y} = -i(\xi \alpha B \sin \alpha y + \beta^2 C \sin \beta y) e^{iy} \]

\[ \frac{\partial u_y}{\partial x} = -i(\xi \alpha B \sin \alpha y - \xi^2 C \sin \beta y) e^{iy} \]

\[ \frac{\partial u_y}{\partial y} = -(\alpha^2 B \cos \alpha y - \beta \xi C \cos \beta y) e^{iy} \]  \hspace{1cm} (4.13)

Hence,

\[ \sigma_{yy} = \left[ -\lambda(\xi^2 B \cos \alpha y + \xi \beta C \cos \beta y) - (\lambda + 2\mu)(\alpha^2 B \cos \alpha y - \beta \xi C \cos \beta y) \right] e^{iy} \]

\[ \sigma_{xy} = -i\mu(2\xi \alpha B \sin \alpha y + \beta^2 C \sin \beta y - \xi^2 C \sin \beta y) e^{iy} \]  \hspace{1cm} (4.14)

Recall boundary conditions \( \sigma_{yx} = \sigma_{xy} = 0 \) at \( y = \pm d \). Imposing these boundary conditions on Equations (4.14) yields

\[ -(\xi^2 - \beta^2) B \cos \alpha d + 2\xi \beta C \cos \beta d = 0 \]

\[ \pm i \left[ -2\xi \alpha B \sin \alpha d + (\xi^2 - \beta^2) C \sin \beta d \right] = 0 \]  \hspace{1cm} (4.15)

or
\[
\begin{bmatrix}
-(\xi^2 - \beta^2) \cos \alpha \theta & 2\xi \beta \cos \beta \theta \\
-2\xi \alpha \sin \alpha \theta & (\xi^2 - \beta^2) \sin \beta \theta 
\end{bmatrix}
\begin{bmatrix}
B \\
C
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\] (4.16)

Equating the determinant of coefficients to zero gives the Rayleigh-Lamb frequency equation for symmetric Lamb waves:

\[
\frac{\tan \beta \theta}{\tan \alpha \theta} = -\frac{4\alpha \beta \xi^2}{(\xi^2 - \beta^2)^2}
\] (4.17)

the constants $B$ and $C$ are determined as the ratio

\[
\frac{B}{C} = \frac{2\xi \beta \cos \beta \theta}{(\xi^2 - \beta^2) \cos \alpha \theta} = \frac{(\xi^2 - \beta^2) \sin \beta \theta}{2\xi \alpha \sin \alpha \theta}
\] (4.18)

4.3 Anti-symmetric motion

Assume displacement anti-symmetric with respect to the plate mid plate:

\[
\begin{align*}
    u_x &= i(\xi A \sin \alpha y - \beta D \sin \beta y)e^{\nu y} \\
    u_y &= (\alpha A \cos \alpha y + \xi D \cos \beta y)e^{\nu y}
\end{align*}
\] (4.19)

The derivatives of $u_x$ and $u_y$ are calculated as

\[
\begin{align*}
    \frac{\partial u_x}{\partial x} &= -(\xi^2 A \sin \alpha y - \xi \beta D \sin \beta y)e^{\nu y} \\
    \frac{\partial u_x}{\partial y} &= i(\xi \alpha A \cos \alpha y - \beta^2 D \cos \beta y)e^{\nu y} \\
    \frac{\partial u_x}{\partial x} &= i(\xi \alpha A \cos \alpha y + \xi^2 D \cos \beta y)e^{\nu y} \\
    \frac{\partial u_x}{\partial y} &= (\alpha^2 A \sin \alpha y + \beta \xi D \sin \beta y)e^{\nu y}
\end{align*}
\] (4.20)

Hence, by using the Navier's equation,
\[ \sigma_{xy} = \left[ -\lambda (\xi^2 A \sin \alpha y - \xi \beta D \sin \beta y) + (\lambda + 2\mu)(\alpha^2 A \sin \alpha y + \beta \xi D \sin \beta y) \right] e^{i\psi} \]

\[ \sigma_{xy} = i\mu (2\xi\alpha A \cos \alpha y - \beta^2 D \cos \beta y + \xi^2 D \cos \beta y) e^{i\psi} \] (4.21)

Recall boundary conditions \( \sigma_{xy} = \sigma_{yy} = 0 \) at \( y = \pm d \). Imposing these boundary conditions on Equations (4.14) yields

\[ -\left( \xi^2 - \beta^2 \right) A \sin \alpha d - 2\xi \beta D \sin \beta d = 0 \]
\[ i\left[ 2\xi\alpha A \cos \alpha d + (\xi^2 - \beta^2) D \cos \beta d \right] = 0 \] (4.22)

or

\[ \begin{bmatrix} -\left( \xi^2 - \beta^2 \right) \sin \alpha d & -2\xi \beta \sin \beta d \\ 2\xi\alpha \cos \alpha d & (\xi^2 - \beta^2) \cos \beta d \end{bmatrix} \begin{bmatrix} A \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \] (4.23)

Equating the determinant of coefficients to zero gives the Rayleigh-Lamb frequency equation for anti-symmetric Lamb waves:

\[ \frac{\tan \beta d}{\tan \alpha d} = \frac{\left( \xi^2 - \beta^2 \right)^2}{4\alpha\beta\xi^2} \] (4.24)

The ratio of constants \( A \), and \( D \)

\[ \frac{A}{D} = \frac{2\xi\beta \sin \beta d}{(\xi^2 - \beta^2) \sin \alpha d} = \frac{(\xi^2 - \beta^2) \cos \beta d}{2\xi\alpha \cos \alpha d} \] (4.25)

4.4 Wave Speed Dispersion Curve

There are two unknown variables in the Rayleigh-Lamb frequency equations: circular frequency \( \omega \) and wave number \( \gamma \). Recall the definition of these two variables, \( \omega = 2\pi f \) and \( \gamma = \frac{\omega}{c} \). Hence, the solutions of Rayleigh-Lamb frequency equations give the
relationship between the frequency and Lamb wave phase velocity, which are the
dispersion curves. Although Rayleigh-Lamb frequency equations have simple forms, the
solution is rather complicated when considering the different numerical conditions.

Rewrite the Rayleigh-Lamb wave equations

\[
\frac{\tan \beta d}{\tan \alpha d} = \left[ \frac{4\alpha \beta \nu^2}{(\xi^2 - \beta^2)^2} \right]^{\pm 1}
\]  

(4.26)

where +1 applies for symmetric modes, while -1 applies for anti-symmetric modes. The
variables \( \alpha \) and \( \beta \) are defined as:

\[
\alpha^2 = \frac{\omega^2}{c_p^2} - \frac{\omega^2}{c_s^2}, \quad \beta^2 = \frac{\omega^2}{c_s^2} - \frac{\omega^2}{c_p^2}
\]  

(4.27)

Depending on the value of \( c \), the terms \( \alpha \) and \( \beta \) may be real, zero or imaginary. These
conditions are described by Graff (1975) as three regions:

**Region I:** \( c < c_s < c_p \).

In this region, \( \alpha^2 = \frac{\omega^2}{c_p^2} - \frac{\omega^2}{c_s^2} < 0 \), and \( \beta^2 = \frac{\omega^2}{c_s^2} - \frac{\omega^2}{c_p^2} < 0 \). Substitution of \( \alpha = i\alpha' \), and
\( \beta = i\beta' \), where \( \alpha'^2 = -\alpha^2 \), \( \beta'^2 = -\beta^2 \), changes the Rayleigh-Lamb frequency equation
to:

\[
\frac{\tanh \beta' d}{\tanh \alpha' d} = \left[ \frac{4\alpha' \beta' \xi^2}{(\xi^2 + \beta'^2)^2} \right]^{\pm 1}
\]  

(4.28)
Region II: \( c_s < c < c_p \).

In this region, \( \alpha^2 = \frac{\omega^2}{c_p^2} - \frac{\omega^2}{c^2} < 0 \), and \( \beta^2 = \frac{\omega^2}{c_s^2} - \frac{\omega^2}{c^2} > 0 \). Substitution of \( \alpha = i\alpha' \) where \( \alpha'^2 = -\alpha^2 \), changes the Rayleigh-Lamb frequency equation to:

\[
\frac{\tan \beta d}{\tanh \alpha' d} = \pm \left[ \frac{4\alpha' \beta \xi^2}{(\xi^2 - \beta^2)^2} \right]^{1/2} \tag{4.29}
\]

Region III: \( c_s < c_p < c \)

In this region, \( \alpha^2 = \frac{\omega^2}{c_p^2} - \frac{\omega^2}{c^2} > 0 \), and \( \beta^2 = \frac{\omega^2}{c_s^2} - \frac{\omega^2}{c^2} > 0 \). The Rayleigh-Lamb frequency equation keeps the form of Equation (4.26).

Rayleigh-Lamb equations can only be solved numerically. Procedures have been developed by various researchers. Viktorov (1967) used an alternative way to express the Rayleigh-Lamb frequency equations, which was more convenient for programming the numerical solution. For symmetric modes,

\[
\frac{\tan(\sqrt{1-\zeta^2} d)}{\tan(\sqrt{\xi^2 - \zeta^2} d)} = \frac{4\zeta^2 \sqrt{1-\zeta^2} \sqrt{\xi^2 - \zeta^2}}{(2\zeta^2 - 1)^2} \tag{4.30}
\]

while for anti-symmetric modes,

\[
\frac{\tan(\sqrt{1-\zeta^2} d)}{\tan(\sqrt{\xi^2 - \zeta^2} d)} = \frac{(2\zeta^2 - 1)^2}{4\zeta^2 \sqrt{1-\zeta^2} \sqrt{\xi^2 - \zeta^2}} \tag{4.31}
\]
where, \( \zeta = \frac{c_s}{c} \), \( \xi = \frac{c_s}{c_p} \), \( \bar{d} = \gamma_s d \), and \( \gamma_s = \frac{\omega}{c_s} \) is the shear wave number at frequency \( \omega \).

In this form, frequency and phase velocity are separated. A program can search the roots \( \zeta_i, (i=0, 1, 2, \ldots) \) for any given \( \bar{d} \).

Rose (1999) made a small variation to the Rayleigh-Lamb frequency equation shown in Equation (4.26). By collecting the terms \( \alpha \) and \( \beta \), the equations take only real values for real or imaginary \( \xi \). The two equations become:

\[
\frac{\tan(\beta d)}{\beta} + \frac{4\xi^2 \alpha \tan(\alpha d)}{(\xi^2 - \beta^2)^2} = 0, \quad \text{symmetric mode} \tag{4.32}
\]

\[
\beta \tan(\beta d) + \frac{(\xi^2 - \beta^2)^2 \tan(\alpha d)}{4\xi^2 \alpha} = 0, \quad \text{anti-symmetric mode}
\]

The roots of the Rayleigh-Lamb equations are commonly plotted on \( fd \) vs. \( c/c_s \) coordinates. The \( fd \) is the product of frequency and half thickness of the plate. By including the thickness factor, plates with different thickness can use same dispersion curves.

For the research presented here, a program was developed using Equation (4.32). The procedures are shown below:

1. Define the desired range of the \( fd \) and \( c/c_s \), such that the \( fd \) vs. \( c/c_s \) plane is defined.

2. Partition the \( fd \) and \( c/c_s \) axes into small steps to create a mesh. Each node on the mesh has a pair of values \( (fd, c/c_s) \).

3. For each of the nodes, calculate the corresponding \( \omega \) and \( c \); plug into Equation (4.27) to obtain \( \alpha \) and \( \beta \). Then, evaluate Equation (4.32)
4. For each \( fd \) value, sweep to vertical line \( fd = \) constant and find sign-change points. These points are close to the roots of the equations. Record the \((fd, c/c_s)\) values of these points.

5. Eliminate the singular points of Equation (4.32), i.e. the points where a jump from +/- infinity to +/- infinity is observed.

6. Use the sign-change points as guess value to find the approximated roots of Equation (4.32). For this step, various iterative root finding algorithms can be used. For example, bisection, Newton-Raphson, etc. Bisection algorithm is used here, because it guarantees convergence. A tolerance is required to define the precision to be achieved.

7. After finding the roots, fit the points with a spline and plot the dispersion curves.

A fine \( fd \) vs. \( c/c_s \) grid is used to increase numerical resolution. (Although finer grid means more calculation, modern CPUs have enough power to finish it in a reasonable time.) In this study, a grid of 200x500 points was used. The iteration precision was set to \( 10^{-8} \). A Pentium 4 (2.4GHz) computer finished the calculation in less than 20 seconds.

Another important property of the Lamb waves is the group velocity dispersion curves. By using the relation

\[
    c_g = c - \lambda \frac{\partial c}{\partial \lambda},
\]

(4.33)
The group velocity, \( c_g \), can be derived from the phase velocity \( c \). To reduce the programming efforts, some manipulations to Equation (4.33) are useful. Following Rose (1999), and using the definition wavelength \( \lambda = \frac{c}{f} \), we write

\[
\frac{\partial c}{\partial \lambda} = \frac{\partial c}{\partial f} \cdot \frac{c}{f} = \frac{\partial c}{f} \left( \frac{1}{\frac{f}{\partial c} - \frac{c}{\partial f}} \right) = \frac{f^2 \partial c}{(f \partial c - c \partial f)}
\]

Hence,

\[
c_g = c - \frac{c}{f} \frac{f^2 \partial c}{(f \partial c - c \partial f)} = c \left( 1 - \frac{f \partial c}{f \partial c - c \partial f} \right) = c \left( -\frac{f \partial c - c \partial f}{c \partial f} \right)^{-1} = c^2 \left( c - f \frac{\partial c}{\partial fd} \right)^{-1}
\]

(4.34)

Equation (4.34) uses the derivative of \( c \) w.r.t. the frequency-thickness product \( fd \). This derivative is calculated from the phase velocity dispersion curve. The numerical derivation is done in a simple way by the finite difference formula:

\[
\frac{\partial c}{\partial (fd)} \approx \frac{\Delta c}{\Delta (fd)}
\]

4.5 Results

Phase velocities of the first several Lamb wave modes, as calculated using the grid, are plotted in Figure 4.3.
Figure 4.3  Numerical solution of the Rayleigh-Lamb frequency equations: (a) symmetric modes; (b) anti-symmetric modes
Figure 4.4  Spline interpolated dispersion curves: (a) symmetric modes; (b) anti-symmetric modes

From the spline interpolated curve, points were taken with small intervals for each mode, and a program was used to calculated the group velocities, which are shown in Figure 4.5.

\[1\] Note: the apparent discontinuities in the lines are an artifact on the plotting software. At higher screen resolution, the lines show absolutely smooth
Figure 4.5 Group velocity dispersion curves: (a) symmetric modes; (b) anti-symmetric modes
CHAPTER 5 THE ASYMPTOTIC BEHAVIOR OF LAMB WAVES

Understanding the structure of the Lamb waves is very important. Lamb waves have many modes across the thickness. In each mode, the displacement and stress distribution patterns changes with frequency and thickness, while the stress-free boundary condition are maintained at upper and lower plate surfaces. All these aspects make the Lamb wave structure complicated compare with conventional P-waves and S-waves. For Non-Destructive Evaluation (NDE) applications, certain types of wave structure are preferred. For example, to detect small defects on the plate surface, it is preferred to have the wave displacement concentrated on the plate surfaces. When using piezoelectric wafer active sensors (PWAS) to excite Lamb waves, it is particularly important to choose a proper wave structure. Because the PWAS are bounded to the surface of the structure, and only coupled through in-plane strain, wave structures having large in-plane strain at the surface are preferred. These examples show the necessity for studying the structure of the Lamb waves.

![Figure 5.1](image)

(a) A plate with thickness $2d$ extends infinitely in x and z direction; (b) free body diagram of a small block on the surface of the plate.
In the following sections, the behavior of the symmetric and anti-symmetric modes Lamb waves will be examined and discussed. To do so, first recall the expressions for Lamb wave displacements and stresses. The displacements are given by:

\[ u_x = i(\gamma B \cos \alpha y + \beta C \cos \beta y)e^{i\psi} + i(\gamma A \sin \alpha y - \beta D \sin \beta y)e^{i\psi} \]
\[ u_y = -(\alpha B \sin \alpha y - \gamma C \sin \beta y)e^{i\psi} + (\alpha A \cos \alpha y + \gamma D \cos \beta y)e^{i\psi} \] (5.1)

where \( \psi = (\gamma x - \omega t) \) is the phase. The derivatives of the particle displacements w.r.t. coordinates \( x \) and \( y \) are

\[ \frac{\partial u_x}{\partial x} = -(\gamma^2 B \cos \alpha y + \gamma \beta C \cos \beta y)e^{i\psi} - (\gamma^2 A \sin \alpha y - \gamma \beta D \sin \beta y)e^{i\psi} \]
\[ \frac{\partial u_x}{\partial y} = -i(\gamma A \sin \alpha y + \beta^2 C \sin \beta y)e^{i\psi} + i(\gamma A \cos \alpha y - \beta^2 D \cos \beta y)e^{i\psi} \]
\[ \frac{\partial u_y}{\partial x} = -i(\gamma A \sin \alpha y - \gamma^2 C \sin \beta y)e^{i\psi} + i(\gamma A \cos \alpha y + \gamma^2 D \cos \beta y)e^{i\psi} \]
\[ \frac{\partial u_y}{\partial y} = -(\alpha^2 B \cos \alpha y - \gamma \beta C \cos \beta y)e^{i\psi} - (\alpha^2 A \sin \alpha y + \gamma \beta D \sin \beta y)e^{i\psi} \] (5.2)

The stresses are given by

\[ \sigma_{xx} = (\lambda + 2\mu) \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) - 2\mu \frac{\partial u_y}{\partial y} \]
\[ \sigma_{yy} = (\lambda + 2\mu) \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) - 2\mu \frac{\partial u_x}{\partial x} \]
\[ \sigma_{xy} = \mu \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \] (5.3)

The stress-free boundary conditions imposed at the upper and lower plate surfaces are \( \sigma_{yy} = 0 \) and \( \sigma_{xy} = 0 \) at \( y = \pm b \).
5.1 Symmetric modes

In the symmetric Lamb wave modes, the displacement parallel to the plate surface, \( u_x \), displays symmetric behavior, \( u_x(-y) = u_x(y) \), while the displacement perpendicular to the plate surface displays anti-symmetric behavior, \( u_y(-y) = -u_y(y) \). Thus, only the terms in \( B \) and \( C \) are retained in Equation (4.10) and the particle motion in symmetric Lamb wave modes becomes

\[
\begin{align*}
    u_x &= i(\gamma B \cos \alpha y + \beta C \cos \beta y)e^{iy} \\
    u_y &= -(\alpha B \sin \alpha y - \gamma C \sin \beta y)e^{iy}
\end{align*}
\]  

(5.4)

Similarly, deduce that the stresses in symmetric Lamb wave modes

\[
\sigma_{xx} = \left[ -\lambda(\alpha^2 B \cos \alpha y - \gamma \beta C \cos \beta y) - (\lambda + 2\mu)(\gamma^2 B \cos \alpha y + \gamma \beta C \cos \beta y) \right] e^{iy} \\
\sigma_{yy} = \left[ -\lambda(\gamma^2 B \cos \alpha y + \gamma \beta C \cos \beta y) - (\lambda + 2\mu)(\alpha^2 B \cos \alpha y - \gamma \beta C \cos \beta y) \right] e^{iy} \\
\sigma_{xy} = -i \mu(2\gamma \alpha B \sin \alpha y + \beta^2 C \sin \beta y - \gamma^2 C \sin \beta y)e^{iy}
\]  

(5.5)

In section 4.2, it has been shown that the imposition of boundary conditions at the plate free surface yields the Rayleigh-Lamb characteristic equations. At a given excitation frequency, solutions of the characteristic equation yields the eigenvalues for the wave numbers, and hence the wave speeds. As already shown, these eigenvalues are invariant with respect to the frequency-thickness product, \( fd \). Figure 4.4 shows a plot of the wave speed vs. the frequency-thickness product for the symmetric Lamb waves (S0, S1, S2 \ldots). Note that, at a given \( fd \) value, several Lamb wave modes may present. For a given material, the wave velocity is affected only by the plate thickness, \( 2d \), and the Lamb wave frequency \( f \). For simplicity of discussion and without losing the generality, make the thickness of the plate a constant value, and discuss the effects of the frequency and mode

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numbers on the Lamb wave structure (across-the-thickness Lamb modes). During the discussion, the low-frequency and high-frequency asymptotic behavior will be considered, as well as behavior at intermediate frequencies. In discussion, assume a 2024-T3 aluminum plate with thickness $2d=1$ mm, with $E = 73.08$ GPa, $\rho = 2768$ kg/m$^3$, $\nu = 0.33$, $c_p=6254.4$ m/s and $c_s=3150.5$ m/s.

![Graph of symmetric modes Lamb wave phase velocity dispersion curves for a 2024-T3 aluminum plate](image)

**Figure 5.2** Symmetric modes Lamb wave phase velocity dispersion curves for a 2024-T3 aluminum plate

5.1.1 Low-frequency long-wavelength behavior

The Lamb wave low-frequency behavior is characterized by the $y$-direction wave length being much longer than the plate half thickness ($\frac{2\pi}{\alpha}, \frac{2\pi}{\beta} \gg d$). Under this assumption, the Lamb wave equations reduce to the equations for in-plane waves within the thin plate theory and the wave speed take the simple expression (Graff, 1975):

$$c_L = \sqrt{\frac{E}{\rho(1-\nu^2)}}$$  \hspace{1cm} (5.6)
For our material, $c_L = 5443 \text{ m/s} = 1.728 \ c_S$. Figure 4.4 indicates that indeed the low-frequency asymptote of the $S0$ wave speed is $c / c_S |_{\alpha \rightarrow \alpha_0} \rightarrow 1.728$. Also apparent in Figure 4.4 is that as $\eta \rightarrow 0$, only the $S0$ Lamb wave mode remains active, the Rayleigh-Lamb equation has only one eigenvalue.

The in-plane wave motion within the classical plate theory is characterized by the assumption that plane sections remain plane and displacements and stresses are uniform across the thickness. This is indeed observed in our study.

For illustration, Figure 5.3 shows the structure of the particle displacements $u_x$ and $u_y$ for the 2024-T3 plate, thickness $2d=1\text{mm}$ at relatively low frequency ($f=100\text{kHz}$) in symmetric Lamb wave $S0$ mode. Figure 5.3(a) shows the displacement vector field

$$\vec{u} = u_x \vec{e}_1 + u_y \vec{e}_2$$

across the thickness and one wavelength along the $x$-axis. The $x$-axis origin is taken at midpoint within the $x$-direction wavelength $\lambda_x$. At this location, $x=0$, and at the extremes, $x = \pm \lambda_x / 2$, the particle displacement is zero. At the fore and aft quarter points, $x = \pm \lambda_x / 4$, the particle displacement is maximum. At the quarter-point locations, the particle displacement is horizontal, i.e. composed only of $x$-direction motion ($u_x = \text{maximum}, u_y = 0$). At the fore quarter point, $x = -\lambda_x / 4$, the particle displacement points toward the right, while at the aft quarter point, it points toward the left. Thus, the maximum difference in horizontal displacement is registered between the two quarter points. Also apparent in Figure 5.3(a) is the Poisson effect. This is represented by the $u_y$ displacement. The $u_y$ displacement is in $\pi/2$ phase w.r.t. the $u_x$ displacement. This means, that $u_y$ magnitude is maximum where $u_y$ is zero at i.e. the extremes, $x = \pm \lambda_y / 2$, and the mid-point, $x = 0$. The phase difference between the $u_x$ and $u_y$ displacements allows the material
to be "squeezed" in and out, in the y-direction, as the waves propagate along the x-direction. Figure 5.3(b) shows the distribution of the displacements through the plate thickness of the plate. Solid line is for the in-plane displacement $u_x$, while the dashed line is for the out-of-plane displacement $u_y$.

![Graph](image)

**Figure 5.3**  Lamb wave S0 mode at $f=100$kHz (a) particle displacement vector (b) amplitude distribution, solid line for $u_x$, dashed line for $u_y$

These plots show that the particle is moving in a symmetric motion, with the displacement almost constant across the thickness. Also note is that the in-plane displacement is much larger than the out-of-plane displacement. This type of particle motion is similar to that found in P-wave (pressure wave), which indicates that the S0 mode Lamb wave at very low frequency will demonstrate similar properties with the P-wave.

5.1.2 Intermediate-frequency intermediate-wavelength behavior

As move away from the low frequency range, the mode shape across the thickness starts to "bow". This process of bowing increases as the frequency increases. Due to "bowing", the surface $u_x$ displacement is less than the mid-plane displacement. As the frequency
increases the surface displacement decrease, becomes zero and then increases in the negative direction. For example, at around $fd=1200 \text{ kHz mm}$, the surface displacement have become negative, though the surface displacement is still much smaller than mid-surface (Figure 5.4).

![Figure 5.4](image)

(a) Lamb wave $S_0$ mode displacement vector and (b) thickness-wise amplitude distribution at $fd=1200 \text{ kHz mm}$.

Further increase $fd$, the surface displacement increases (in absolute value) while the mid-plane displacement decreases. At $fd=1500 \text{ kHz mm}$, the surface displacement amplitude equals the mid-plane displacement, while being of opposite sign (Figure 5.5).

![Figure 5.5](image)

(a) Lamb wave $S_0$ mode displacement vector and (b) thickness-wise amplitude distribution at $fd=1500 \text{ kHz mm}$. 

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As the $fd$ value increases even further, the wave activity starts to concentrate in the upper and lower surfaces. At about $fd=3100$ kHz mm, $u_x$ displacement is almost flat over the inner 80% part of the thickness, while most of the activity happens in the upper and lower 10% parts (Figure 5.6).

![Image of Lamb wave S0 mode displacement vector and thickness-wise amplitude distribution at $fd=3100$ kHz mm.](image)

**Figure 5.6**  
(a) Lamb wave S0 mode displacement vector and (b) thickness-wise amplitude distribution at $fd=3100$ kHz mm.

Beyond $fd=3100$kHz:mm, the concentration of the wave near to the surface increases even more. At $fd=5000$ kHz mm, wave activity is only observed near the surface, and is practically zero for the rest (Figure 5.7). Beyond this point, the high-frequency low-wavelength limit was reached, which will be discussed in a later section.

![Image of Lamb wave S0 mode displacement vector and thickness-wise amplitude distribution at $fd=5000$ kHz mm.](image)

**Figure 5.7**  
(a) Lamb wave S0 mode displacement vector and (b) thickness-wise amplitude distribution at $fd=5000$ kHz mm.
5.1.3 Effect of wave number

The switching from one Lamb S mode to another can have a profound effect on the wave amplitude distribution across the thickness. For example, at $fd=1500$ kHz mm, the S0 mode shape shows a balanced distribution of the $u_z$ displacement between the free surfaces and the mid-plane. As shown in Figure 5.8a, the mid-plane deflection and the surface displacement are about the same amplitude, although of different signs. If it changes from the S0 mode to the S1 mode, the situation changes dramatically (Figure 5.8b).

![Figure 5.8](image)

**Figure 5.8** Lamb wave displacement vector and thickness-wise amplitude distribution at $fd=1500$ kHz mm. (a) S0 mode, (b) S1 mode
In the S1 mode, at the same $fd=1500$ kHz:mm, the upper and lower surface displacements have a magnitude almost 5 times larger than the mid-plane displacement, thus indicating that, at this $fd$ value, the S1 activity is much larger at the free surfaces than the S0 activity.

Figure 5.9  (a)Lamb wave S1 mode displacement vector and (b) thickness-wise amplitude distribution at $fd=1800$ kHz mm.

Increase the $fd$ value from 1500 to 1800 kHz:mm, the mid-plane displacement of the S1 mode goes through zero, while the surface displacement goes through a maximum. At $fd=1800$ kHz:mm the thickness of the plate is almost exactly equal to the wave length of the S1 mode in the thickness direction. A “perfect” sinusoid fits between the upper and lower free surfaces (Figure 5.9).

Figure 5.10  (a)Lamb wave S1 mode displacement vector and (b) thickness-wise amplitude distribution at $fd=2000$ kHz mm.
This situation of zero mid-plane displacement persists till $fd=2000$ kHz·mm (Figure 5.10). Beyond this value, the mid-plane displacement starts to increase while the surface displacement decreases. The decrease in the surface displacement is due to the shortening of the thickness-wise wavelength, which makes the surface plane intersect the displacement curve beyond the maximum of the sinusoidal shape.

![Graphs](image)

**Figure 5.11** (a) Lamb wave S1 mode displacement vector and (b) thickness-wise amplitude distribution at $fd=2900$ kHz·mm.

This phenomena continues till about $fd=2900$ kHz·mm (Figure 5.11). At this $fd$ value, the $u_x$ displacement amplitudes at the mid-plane and the upper and lower surfaces are again almost equal, although of opposite signs.

![Graphs](image)

**Figure 5.12** (a) Lamb wave S1 mode displacement vector and (b) thickness-wise amplitude distribution at $fd=3500$ kHz·mm.

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Continue increasing the $fd$ value, the wavelength in the thickness direction keeps decreasing, such that at about $fd=3500$ kHz·mm, the surface displacement reaches the zero point of the sinusoid, whereas the mid-plane displacement reaches a maximum (Figure 5.12).

![Figure 5.12](image)

Figure 5.13  (a)Lamb wave S1 mode displacement vector and (b) thickness-wise amplitude distribution at $fd=6000$ kHz mm.

Beyond this point, the surface displacement will increase in the positive direction, while the mid-plane displacement will decrease. At about $fd=6000$ kHz mm (Figure 5.13), the surface displacement and the mid-plane displacement have about the same values.

![Figure 5.13](image)
Figure 5.14  Lamb wave displacement vector and thickness-wise amplitude distribution at \(fd=3500\ kHz\) mm. (a) S1 mode, (b) S2 mode

To see the effects of switching the mode number to an even higher value, move back to \(fd=3500\ kHz\) mm, where the upper and lower surface displacements in the S1 mode were zero, while the mid-plane displacement was a maximum (Figure 5.14a). Switching from S1 to S2 mode, it can be observed that the surface displacement becomes large and almost equal to the mid-plane displacement (Figure 5.14b).

Figure 5.15  (a)Lamb wave S2 mode displacement vector and (b) thickness-wise amplitude distribution at \(fd=3700\ kHz\) mm.

Travel along the S2 dispersion curve, at the point \(fd=3700kHz\cdot mm\), where the upper and lower surface displacements are equal to the mid-plane displacement. This situation correspond to two thickness-wise wavelength fitting exactly into the plate thickness (Figure 5.15)
Beyond $fd=3700$ kHz·mm, the wavelength in the thickness direction continues to decrease, and hence the surface displacement continue to decrease. This happens because the sinusoidal shape is intersected by the surface planes at a position beyond its maximum. It was also observed that the relative magnitude between the surface and mid-plane displacements also decrease indicating that the wave motion starts to become concentrated toward the upper and lower surfaces. In addition, a new feature appears, i.e. the negative displacement at the quarter point across the thickness (Figure 5.16 at $fd=5000$ kHz·mm).

![Figure 5.16](image)

Figure 5.16 (a) Lamb wave S2 mode displacement vector and (b) thickness-wise amplitude distribution at $fd=5000$ kHz·mm.

Further increasing the $fd$ value, the quarter point amplitudes also increase while the mid-plane displacement decrease. At about $fd=5200$ kHz·mm, the quarter-point and mid-plane displacements are almost equal. However, they are less than the amplitudes in the upper and lower 20% of the thickness (Figure 5.17).
Figure 5.17  (a)Lamb wave S2 mode displacement vector and (b) thickness-wise amplitude distribution at \( f_d = 5200 \text{ kHz mm.} \)

Figure 5.18  (a)Lamb wave S2 mode displacement vector and (b) thickness-wise amplitude distribution at \( f_d = 5450 \text{ kHz mm.} \)

Beyond \( f_d = 5200 \text{ kHz mm,} \) the process of wave energy migration from center towards the outer surfaces is becoming increasingly pregnant. At \( f_d = 5450 \text{ kHz mm,} \) the thickness direction sinusoid has degenerated into a shape which is sinusoidal only in the inner 50% of the thickness, while monotonically increasing in the upper and lower 25% of the thickness. In fact, at this \( f_d \) value, the maximum amplitude in the inner 50% of the thickness is less than 1/5 of the amplitudes of the free surfaces (Figure 5.18).
Figure 5.19  (a)Lamb wave S2 mode displacement vector and (b) thickness-wise amplitude distribution at \(fd=5490 \text{ kHz mm}\).

This ratio decreases rapidly as the \(fd\) product increases, such that at \(fd=5490 \text{ kHz mm}\), the energy is almost entirely contained in the outer 25% portions of the thickness. At \(fd=5500 \text{ kHz mm}\), the limit of the intermediate-frequency, intermediate-wavelength is reached, and beyond this value, the high-frequency, low-wavelength situation is experienced.

5.1.4  High frequency-short wavelength behavior

Figure 5.7 shows the displacement vectors and distribution curves at 5000 kHz. It is noticed that the displacement amplitude decreases rapidly with the increment of the thickness. When the Lamb wave frequency increases to a very high value, the wavelength becomes very small w.r.t. the thickness of the plate, and the Lamb wave degrades to the Rayleigh wave (surface wave). In this type of wave, the particle displacement amplitude decrease exponentially in depth, so only the particles close to the surface participate in the wave motion.
5.1.5 Change of relative amplitude of \( u_x \) and \( u_y \) components with \( fd \) value

It is noticed that the distribution of the in-plate and out-of-plate displacement change with the frequency. It also should be considered that the phase velocity is determined by both the frequency \( f \) and thickness \( 2d \). Thus the relative amplitude of in-plate displacement \( (u_x) \) and out-of-plate displacement \( (u_y) \) changes with the frequency-thickness product \( (fd) \) value.

A general observation is that at low \( fd \) value, \( u_x \) is the dominant component, which is almost constant through the thickness; the wave structure is simple and very close to that of P-wave. When \( fd \) value goes higher, \( u_y \) component increases and \( u_x \) gets concentrated in the center of the plate, and the wave structure is getting complicated. At very high \( fd \) value, the displacement decreases rapidly in thickness direction, and the Lamb wave degrade to Rayleigh waves on both surfaces.

It is interesting to observe the change of the surface displacement. One of the practices of non-destructive evaluation is to detect surface defects. In this type of applications, larger in-plate particle motion on the surfaces is preferable. The benefits of having large surface in-plate displacement are: (1) increasing the propagation distance along the structure; (2) increasing the interaction between the wave and the surface defects; (3) increasing the efficiency of surface mounted transducers (e.g. PWAS).

The in-plate and out-of-plate displacement amplitude is plotted against the \( fd \) product for a 20240-T3 aluminum plate in Figure 5.20.
In-plate ($u_x$) and out-of-plane ($u_y$) component of the surface displacement for S0 mode Lamb wave. Solid line is for $u_x$, and dashed line is for $u_y$. (Material is 2024-T3 aluminum)

The $fd$ value are plotted in the typical working frequency range of PWAS transducers. It is noticed that the in-plate displacement have a peak value at $fd=600$ kHz mm. Suppose the thickness ($2d$) of the plate is 1mm, the frequency will be 1200 kHz. This frequency then can be used as a candidate of working points.

5.2 Anti-symmetric modes

In the anti-symmetric Lamb wave modes, the displacement parallel to the plate surface displays anti-symmetric behavior, $u_x(-y) = -u_x(y)$, while the displacement perpendicular to the plate surface displays symmetric behavior, $u_y(-y) = u_y(y)$. Thus, only the terms in $A$ and $D$ are retained in Equation (5.7) and the particle motion becomes

$$u_x = i(\gamma A \sin \alpha y - \beta D \sin \beta y)e^{iy}$$
$$u_y = (\alpha A \cos \alpha y + \gamma D \cos \beta y)e^{iy}$$

(5.7)

Similarly, deduce that the stresses in symmetric Lamb wave modes

$$\sigma_{xx} = \left[-\lambda(\alpha^2 A \sin \alpha y + \gamma \beta D \sin \beta y) - (\lambda + 2\mu)(\gamma^2 A \sin \alpha y - \gamma \beta D \sin \beta y)\right]e^{iy}$$
$$\sigma_{yy} = \left[-\lambda(\gamma^2 A \sin \alpha y - \gamma \beta D \sin \beta y) - (\lambda + 2\mu)(\alpha^2 A \sin \alpha y + \gamma \beta D \sin \beta y)\right]e^{iy}$$
$$\sigma_{xy} = i\mu(2\gamma \alpha A \cos \alpha y - \beta^2 D \cos \beta y + \gamma^2 D \cos \beta y)e^{iy}$$

(5.8)
In section 4.3, it has been shown that the imposition of boundary conditions at the plate free surface yields the Rayleigh-Lamb characteristic equations. At a given excitation frequency, solutions of the characteristic equation yields the eigenvalues for the wave numbers, and hence the wave speeds. As already shown, these eigenvalues are invariant with respect to the frequency-thickness product, $fd$. Figure 5.21 shows a plot of the wave speed versus the frequency-thickness product for the anti-symmetric Lamb waves ($A_0$, $A_1$, $A_2$ ...). Note that, at a given $fd$ value, several Lamb wave modes may be present. For a given material, the wave velocity is affected only by the plate thickness, $2d$, and the Lamb wave frequency $f$. For simplicity of discussion and without losing generality, make the thickness of the plate a constant value, and discuss the effects of the frequency and mode numbers on the Lamb wave structure (across-the-thickness Lamb modes). During our discussion, the low-frequency and high-frequency asymptotic behavior will be considered, as well as behavior at intermediate frequencies. In the discussion, assume a 2024-T3 aluminum plate with thickness $2d=1$ mm, with $E = 73.08$ GPa, $\rho = 2768$ kg/m$^3$, $\nu = 0.33$, $c_p=6254.4$ m/s and $c_S=3150.5$ m/s
5.2.1 Low-frequency long-wavelength behavior

The Lamb wave low-frequency behavior is characterized by the $y$-direction wavelength being much longer than the plate half thickness ($\frac{2\pi}{\alpha}, \frac{2\pi}{\beta} \gg d$). Under this assumption, the Lamb wave equations reduce to the equations for flexural waves within the classical plate theory and the wave speed take the simple expression (Graff, 1975):

$$c_L = \gamma d \sqrt{\frac{E}{3\rho(1-\nu^2)}} \quad (5.9)$$

The flexural wave motion within the classical plate theory is characterized by the assumption that plane sections remain plane and displacements and stresses vary linearly across the thickness. Equation (5.9) predicts the wave speed goes to zero at $fd = 0$. This is indeed observed in our study.
Figure 5.22    Lamb wave A0 mode at f=100kHz (a) particle displacement vector (b) amplitude
distribution, solid line for \( u_x \), dashed line for \( u_y \) (2024-T3 aluminum plate)

For illustration, Figure 5.22 shows the structure of the particle displacements \( u_x \) and \( u_y \) in
the anti-symmetric Lamb wave A0 mode at a relatively low frequency \( (f=100\text{kHz}) \).

Figure 5.22(a) shows the displacement vector field \( \vec{u} = u_x \vec{e}_1 + u_y \vec{e}_2 \) across the thickness
and over two wavelengths along the \( x \)-axis. The \( x \)-axis origin is taken at midpoint within
the \( x \)-direction wavelength \( \lambda_x \). At this location, \( x=0 \), and at the extremes, \( x = \pm \lambda_x / 2 \), the
particle displacement is maximum. At the fore and aft quarter points, \( x = \pm \lambda_x / 4 \), the
particle displacement is zero. At the quarter-point locations, the particle displacement is
vertical, i.e. composed only of \( y \)-direction motion \( (u_y = \text{maximum, } u_x = 0) \). At the fore
quarter point, \( x = -\lambda_x / 4 \), the particle displacement points toward +\( y \) direction, while at
the aft quarter point, it points toward −\( y \) direction. Thus, the maximum difference in
horizontal displacement is registered between the two quarter points. Figure 5.22(b)
shows the distribution of the displacements through the plate thickness of the plate. Solid
line is for the in-plate displacement \( u_x \), while the dashed line is for the out-of-plate
displacement \( u_y \).
These plots show that the particle is moving in an anti-symmetric motion, with the \( u_z \) displacement varying linearly across the thickness and the \( u_y \) displacement almost constant across the thickness. Also note that the out-of-plane displacement \( u_y \) is two times larger than the in-plane displacement \( u_x \). This type of particle motion is similar to that found in flexural wave, which indicates that the A0 mode Lamb wave at very low frequency will demonstrate similar properties with the classical flexural wave.

5.2.2 Intermediate-frequency intermediate-wavelength behavior

Moving away from the low frequency range, both the in-plane and out-of-plane displacements change. For example, at around \( fd=700 \text{ kHz-mm} \), (Figure 5.23), the in-plane displacement \( u_x \) is no longer linearly related to the \( y \) position, while the out-of-plane displacement \( u_y \) is no longer constant across the thickness. Further increase \( fd \), the surface displacement increases (in absolute value) faster than the mid-plane displacement, so the mode shape across the thickness starts to “bow”. This process of “bowing” increases as the frequency increases. At \( fd=1200 \text{ kHz-mm} \), the mid-surface \( u_y \) displacement starts to show a smaller amplitude than that on the surface (Figure 5.24).

\[ \text{Figure 5.23 (a) Lamb wave A0 mode displacement vector and (b) thickness-wise amplitude distribution at } fd=700 \text{ kHz mm.} \]
Figure 5.24 (a) Lamb wave A0 mode displacement vector and (b) thickness-wise amplitude distribution at fd=1200 kHz mm.

At this $fd$ value, the $u_x$ displacement is almost zero over the inner 60% part of the thickness, while most of the activity happens in the upper and lower 40% parts.

As the $fd$ value increases even further, the $u_y$ displacement crosses zero and changes sign at a small depth below the surface. Both $u_x$ and $u_y$ become more concentrated on the surface. At about $fd=3000$ kHz mm, the $u_x$ displacement is almost zero over the inner 80% part of the thickness, while most of the activity happens in the upper and lower 20% parts; the $u_y$ displacement on the surface is almost twice as large as on the mid-surface (Figure 5.25).

Figure 5.25 (a) Lamb wave A0 mode displacement vector and (b) thickness-wise amplitude distribution at fd=3000 kHz mm.
Beyond \( f/d = 3000 \text{ kHz:mm} \), the concentration of the wave near to the surface increases even more. At \( f/d = 10,000 \text{ kHz:mm} \), wave activity is only observed near the surface, and is practically zero elsewhere (Figure 5.26). Beyond this point, the high-frequency low-wavelength limit is reached, which will be discussed in a later section.

![Figure 5.26](a) Lamb wave A0 mode displacement vector and (b) thickness-wise amplitude distribution at \( f/d = 10000 \text{ kHz:mm} \).

5.2.3 Effect of wave mode number

The switching from one Lamb S mode to another can have a profound effect on the wave amplitude distribution across the thickness. For example, at \( f/d = 1100 \text{ kHz:mm} \), the A0 mode shape shows an almost even distribution of the \( u_r \) displacement, while the \( u_r \) displacement is started to show some degrees of concentration on the surface (Figure 5.27a). If it changes from the A0 mode to the A1 mode, the situation changes dramatically (Figure 5.27b).
Figure 5.27  Lamb wave displacement vector and thickness-wise amplitude distribution at $fd=1100$ kHz mm. (a) A0 mode, (b) A1 mode

In the A1 mode, at the same $fd=1100$ kHz:mm, the $u_y$ displacement is showing a clear "bow", where the surface displacement is large while the mid-surface displacement is zero. The $u_x$ displacements are large on the upper and lower surfaces and showing a half-sinusoidal pattern. Also, it is zero at the mid point.

At $fd=2250$ kHz:mm, the $u_x$ displacement has a sinusoidal-like pattern, with the upper, lower and quarter-points having maximum displacements (Figure 5.28).
Figure 5.28 (a) Lamb wave A1 mode displacement vector and (b) thickness-wise amplitude distribution at \( f_d = 2250 \text{ kHz mm} \).

When increase the frequency, the quarter-points displacements are getting larger, and the surface displacements decrease. At 2600 kHz mm (Figure 5.29), the quarter-points displacements are about twice as large as the upper and lower surface displacements.

Figure 5.29 (a) Lamb wave A1 mode displacement vector and (b) thickness-wise amplitude distribution at \( f_d = 2600 \text{ kHz mm} \).

To see the effects of switching the mode number to an even higher value, switch from A1 to A2 mode. It was observed that the surface displacement becomes large and almost twice as large as the quarter-point displacement (Figure 5.30b).
Figure 5.30  (a)Lamb wave S2 mode displacement vector and (b) thickness-wise amplitude distribution at \( fd=2600 \) kHz mm.

Travel along the S2 dispersion curve, at point \( fd=4000 \) kHz mm, where the wave structure is very complicate. The maximum displacements happen at about 1/6 thickness from the upper and lower surface. Upper and lower surface displacement are about the same as those at 1/3 thickness, and they are also large. In general, when frequency goes higher in the intermediate frequency range, the wave structure gets more complicate and multiple large displacements appear through the thickness (Figure 5.31)

Figure 5.31  (a)Lamb wave S2 mode displacement vector and (b) thickness-wise amplitude distribution at \( fd=4000 \) kHz mm.

5.2.4 High-frequency short-wavelength behavior

Figure 5.26 showed the displacement vectors and distribution curves at \( fd=10,000 \) kHz:mm. It is noticed that the displacement amplitude decreases rapidly with the
increment of the thickness. When the Lamb wave frequency increases to a very high value, the wavelength becomes very small with respect to the thickness of the plate, and the $A_0$ Lamb wave degrades to the Rayleigh wave. In this type of wave, the particle displacement amplitude decreases exponentially in depth, so only the particles close to the surface participate in the wave motion.

5.2.5 Change of relative amplitude of $u_x$ and $u_y$ components with $fd$ value

It is noticed that the distribution of the in-plate and out-of-plate displacement change with the frequency. It also should be considered that the phase velocity is determined by both the frequency ($f$) and thickness ($2d$). Thus the relative amplitude of in-plate displacement ($u_x$) and out-of-plate displacement ($u_y$) changes with the frequency-thickness product ($fd$) value.

A general observation is that at low $fd$ value, $u_y$ is the dominant component, which is almost constant through the thickness; the wave structure is simple and very close to that of shear wave. When $fd$ value goes higher, $u_x$ component increases in relative amplitude, and the wave structure is getting complicated. As $fd$ increases further, $u_x$ and $u_y$ become concentrated to the surface of the plate. At very high $fd$ values, the displacements decrease rapidly with the distance away from the surface, and the Lamb wave becomes Rayleigh waves on both upper and lower surfaces.
CHAPTER 6 CIRCULAR CRESTED LAMB WAVES

Straight crested Lamb waves' behavior has been covered in *Wave motion in elastic solids* (Graff, 1975) in detail. Another variation of wave in plates which is more commonly seen in practice is circular crested waves. Goodman (1952) investigated circular crested waves, and found that the Rayleigh-Lamb frequency equations are still valid in this case and the straight crested Lamb wave behavior is the limit of circular crested waves. Graff (1975) introduced Goodman's work briefly in his book. The study of the behavior of this variation of waves is important when circular crested waves are used in applications. Among all the properties, wave phase velocity and particle displacement on the surface are essential. Once these are obtained, other properties can be derived.

6.1 PROBLEM DEFINITION

![Diagram](image)

Figure 6.1 A plate with thickness $2d$, extends infinitely in $r$ direction.

The problem on hand is a plate with traction-free surfaces under axisymmetric loading. Following Goodman (1952), cylindrical coordinates is used. Consider a plate with thickness $2d$, and axisymmetric conditions hold, so a circular wavefront propagates in the $r$ direction. (Figure 6.1).
6.2 Equations and Derivations

The traction-free surface condition can be described as: \( \sigma_{zn} = \sigma_{zr} = \sigma_{z\theta} = 0 \) at \( z = \pm d \).

The axisymmetric condition implies \( \frac{\partial}{\partial \theta} = 0 \) in Cartesian coordinates, plane strain condition can be described as \( z \)-invariant. Similar to that, the condition we are dealing in cylindrical coordinates can be described as \( \theta \)-invariant. The governing equations in terms of displacement are the Navier's equations. All the \( \frac{\partial}{\partial \theta} \) terms vanish and only two of the equations are non-trivial. These two equations are (Goodman, 1952, pp.66)

\[
\begin{align*}
(\lambda + \mu) \frac{\partial \Delta}{\partial r} + \mu \nabla^2 u_r - \mu \frac{u_r}{r^2} &= \rho u_r \\
(\lambda + \mu) \frac{\partial \Delta}{\partial z} + \mu \nabla^2 u_z &= \rho u_z
\end{align*}
\]  
\( (6.1) \)

A similar condition was covered by Timoshenko (1959), in analyzing the static deflection of a circular plate on elastic foundation. By analogy with the static solutions, the dilatation \( \Delta \) is assumed to have the form:

\[
\Delta = A \frac{\mu}{\lambda + \mu} J_0(\xi r) e^{i\omega t} \begin{cases} \cosh \alpha z & \text{(axial)} \\ \sinh \alpha z & \text{(flexural)} \end{cases}
\]  
\( (6.2) \)

The former will correspond to axial vibrations in which the mid-plane is stretched (or compressed) and motions on either side of it are symmetrical. The latter form corresponds to flexural vibrations in which the mid-plane is un-stretched and motions on either side of it are anti-symmetrical.

We also assumed the two displacements \( u_r \) and \( u_z \) in the forms
\[ u_r = f_1(z) J_1(\xi r)e^{i\omega t} \]
\[ u_z = f_2(z) J_0(\xi r)e^{i\omega t} \]  

(6.3)

Their 1st order derivatives are

\[ \frac{\partial}{\partial r} u_r = f_1'(z) J_1(\xi r)e^{i\omega t} \]
\[ \frac{\partial}{\partial z} u_r = -f_2(z)\xi J_1(\xi r)e^{i\omega t} \]
\[ \frac{\partial}{\partial r} u_z = f_1(z) J_0(\xi r)e^{i\omega t} \]
\[ \frac{\partial}{\partial z} u_z = f_2'(z) J_0(\xi r)e^{i\omega t} \]  

(6.4)

and the 2nd order derivatives are

\[ \frac{\partial^2}{\partial r^2} u_r = f_1(z) \xi^2 \left( -\xi J_1(\xi r) + \frac{2}{\xi r^2} J_0(\xi r) - \frac{1}{r} J_1(\xi r) \right) e^{i\omega t} \]
\[ \frac{\partial^2}{\partial z^2} u_r = f_1(z) J_1(\xi r)e^{i\omega t} \]
\[ \frac{\partial^2}{\partial z^2} u_z = f_2(z) J_0(\xi r)e^{i\omega t} \]
\[ \frac{\partial^2}{\partial r^2} u_z = -f_2(z)\xi \xi^2 \left( J_0(\xi r) - \frac{1}{\xi r} J_1(\xi r) \right) e^{i\omega t} \]  

(6.5)

In this derivation, we used the standard results: (Beyer, 1987 pp 444)

\[ \frac{\partial}{\partial z} J_0(z) = -J_1(z) \]  

\[ \frac{\partial}{\partial r} J_1(z) = J_0(z) - \frac{1}{z} J_1(z) \]  

(6.6)

\[ \text{Derivative of } J_1 \text{ is derived from the recurrence formulas (Beyer, 1987 pp 444)} \]
\[ nJ_n(x) + xJ'_n(x) = xJ_{n-1}(x), \text{ here, } J_1(z) + zJ'_1(z) = zJ_0(z) \Rightarrow J'_1(z) = J_0(z) - \frac{1}{z} J_1(z). \]
Hence,

\[
\frac{\partial}{\partial r} J_0 (\xi r) = \frac{\partial (\xi r)}{\partial r} \frac{\partial}{\partial (\xi r)} J_0 (\xi r) = -\xi J_1 (\xi r)
\]

\[
\frac{\partial}{\partial r} J_1 (\xi r) = \frac{\partial (\xi r)}{\partial r} \frac{\partial}{\partial (\xi r)} J_1 (\xi r) = \xi \left( J_0 (\xi r) - \frac{1}{\xi r} J_1 (\xi r) \right)
\]

(6.7)

In cylindrical coordinates, the dilatation \( \Delta \) is defined as

\[
\Delta = \frac{\partial u_r}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}
\]

(6.8)

Substituting Equations (6.3) and (6.4) into Equation (6.8) yields:

\[
\Delta = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}
\]

\[
= f_1(z)\xi \left( J_0 (\xi r) - \frac{1}{\xi r} J_1 (\xi r) \right) e^{i\omega t} + \frac{1}{r} f_1(z) J_1 (\xi r) e^{i\omega t} + f_2(z) J_0 (\xi r) e^{i\omega t}
\]

(6.9)

\[
= \left[ \xi f_1(z) + f_2'(z) \right] J_0 (\xi r) e^{i\omega t}
\]

Equation (6.9) indicates that, in order to satisfy Equation (6.2), it is necessary to have

\[
\xi f_1(z) + f_2'(z) = A \frac{\mu}{\lambda + \mu} \begin{cases} \cosh \alpha z & \text{(axial)} \\ \sinh \alpha z & \text{(flexural)} \end{cases}
\]

(6.10)

Using Equation (6.2) and (6.3), the terms of the Navier’s Equations can be derived as follows.

The Laplacian of a scalar in cylindrical coordinates is defined as

\[
\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2}
\]

(6.11)
The Laplacian of $u_r$ is

$$
\nabla^2 u_r = \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2}
$$

$$
= \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} + \frac{\partial^2 u_r}{\partial z^2}
$$

$$
= f_1(z) \xi \left( -\xi J_1(\xi r) + \frac{2}{\xi r^2} J_1(\xi r) - \frac{1}{r} J_0(\xi r) \right) e^{i\omega t} \tag{6.12}
$$

$$
+ \frac{1}{r} f_1(z) \xi \left( J_0(\xi r) - \frac{1}{\xi r} J_1(\xi r) \right) + f_1''(z) J_1(\xi r) e^{i\omega t}
$$

$$
= \left[ f_1''(z) - f_1(z) \left( \frac{\xi^2}{r^2} - \frac{1}{r^2} \right) \right] J_1(\xi r) e^{i\omega t}
$$

The Laplacian of $u_z$ is

$$
\nabla^2 u_z = \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} \frac{\partial u_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}
$$

$$
= \left[ -f_2(z) \xi J_0(\xi r) + f_2''(z) J_0(\xi r) \right] e^{i\omega t} \tag{6.13}
$$

$$
= J_0(\xi r) e^{i\omega t} \left[ -f_2(z) \xi^2 + f_2''(z) \right]
$$

The 1st order derivatives of the dilatation $\Delta$ are

$$
\frac{\partial}{\partial r} \Delta = -\xi J_1(\xi r) A \frac{\mu}{\lambda + \mu} g(\alpha z) e^{i\omega t} \tag{6.14}
$$

$$
\frac{\partial}{\partial z} \Delta = \alpha J_0(\xi r) A \frac{\mu}{\lambda + \mu} g'(\alpha z) e^{i\omega t}
$$

where

$$
g(\alpha z) = \begin{cases} 
\cosh \alpha z & \text{(axial)} \\
\sinh \alpha z & \text{(flexural)} 
\end{cases} \tag{6.15}
$$

Substitution of Equations (6.3), (6.12), (6.13), and (6.14) into the Navier's Equations (6.1) yields:
\[(\lambda + \mu) \frac{\partial \Delta}{\partial \xi} + \mu \nabla^2 u_r - \frac{u_r}{r^2} = \rho \ddot{u}_r \]
\[-\lambda \xi J_1(\xi r) Ag(\alpha z)e^{i\omega t} + \mu J_1(\xi r)e^{i\omega t} \left( f'_1(z) - \xi^2 f_1(z) \right) \]
\[= \rho \left( -\omega^2 \right) f_1(z) J_1(\xi r)e^{i\omega t} \]
\[f'_1(z) - f_1(z) \left( \xi^2 - \frac{\rho \omega^2}{\mu} \right) = A \xi g(\alpha z) \]

\[(\lambda + \mu) \frac{\partial \Delta}{\partial \xi} + \mu \nabla^2 u_z = \rho \ddot{u}_z \]
\[\alpha \mu Ag'(\alpha z) J_0(\xi r)e^{i\omega t} + \mu \left[ f'_2(z) - f_2(z) \xi^2 \right] J_0(\xi r)e^{i\omega t} \]
\[= \rho \left( -\omega^2 \right) f_2(z) J_0(\xi r)e^{i\omega t} \]
\[f'_2(z) - f_2(z) \left( \xi^2 - \frac{\rho \omega^2}{\mu} \right) = -A \alpha g'(\alpha z) \]

Introduce
\[\beta^2 = \xi^2 - \frac{\rho \omega^2}{\mu} = \xi^2 - \frac{\omega^2}{c_i^2} \]

\[\alpha^2 = \xi^2 - \frac{\rho}{\lambda + 2\mu} \omega^2 = \xi^2 - \frac{\omega^2}{c_p^2} \]

Hence, Equations (6.16) and (6.17) can be written as
\[f'_1(z) - f_1(z) \beta^2 = A \xi g(\alpha z) \]
\[f'_2(z) - f_2(z) \beta^2 = -A \alpha g'(\alpha z) \]

By solving Equation (6.20) and Equation (6.10), the two functions, \(f_1(z)\) and \(f_2(z)\), can be determined. Hence the displacement functions can be also determined. For axial vibration case
\[f'_1(z) - f_1(z) \beta^2 = A \xi \cosh(\alpha z) \]

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\[ f'_1(z) - f_1(z)\beta^2 = -A\alpha \sinh(\alpha z) \quad (6.21b) \]

The general solution of the homogeneous differential equation \( f''_1(z) + \beta^2 f_1(z) = 0 \) is

\[ f_{1c}(z) = C_1 e^{\beta z} + C_2 e^{-\beta z} = C_1 \sinh(\beta z) + C_2 \cosh(\beta z) \quad (6.22) \]

Assume the particular solution is \( f_{1p}(z) = D \cosh(\alpha z) \). Then \( f''_1(z) = \alpha^2 f_{1p}(z) \). Upon substitution into Equation (6.21),

\[ f_{1p}(z) = A \frac{\xi}{\alpha^2 - \beta^2} \cosh(\alpha z) \quad (6.23) \]

Hence,

\[ f_1(z) = f_{1c}(z) + f_{1p}(z) = C_1 \sinh(\beta z) + C_2 \cosh(\beta z) + A \frac{\xi}{\alpha^2 - \beta^2} \cosh(\alpha z) \quad (6.24) \]

To satisfy the axial vibration condition, we must have \( f_1(z) = f_1(-z) \). This yields \( C_1 = 0 \), hence the solution for \( f_1(z) \) is

\[ f_1(z) = C_2 \cosh(\beta z) + A \frac{\xi}{\alpha^2 - \beta^2} \cosh(\alpha z) \quad (6.25) \]

Similarly, the general solution of \( f''_2(z) + \beta^2 f_2(z) = 0 \) is

\[ f_{2c}(z) = \tilde{E}_1 e^{\beta z} + \tilde{E}_2 e^{-\beta z} = E_1 \sinh(\beta z) + E_2 \cosh(\beta z) \quad (6.26) \]

Assume the particular solution is \( f_{2p}(z) = F \sinh(\alpha z) \). Then \( f''_2(z) = \alpha^2 f_{2p}(z) \), plug in to Equation (21.b), solving for

\[ f_{2p}(z) = -A \frac{\alpha}{\alpha^2 - \beta^2} \sinh(\alpha z) \quad (6.27) \]
Hence,

\[ f_2(z) = f_{2c}(z) + f_{2p}(z) = E_1 \sinh(\beta z) + E_2 \cosh(\beta z) - A \frac{\alpha}{\alpha^2 - \beta^2} \sinh(\alpha z) \quad (6.28) \]

To satisfy the axial vibration condition, must have \( f_2(z) = -f_2(-z) \). This yields \( E_2 = 0 \).

Hence, the solution for \( f_2(z) \) is

\[ f_2(z) = E_1 \sinh(\beta z) - A \frac{\alpha}{\alpha^2 - \beta^2} \sinh(\alpha z) \quad (6.29) \]

To obtain the constants \( C_2 \) and \( E_1 \), substitute Equations (6.25) and (6.29) into Equation (6.10):

\[ \xi f_1(z) + f_2'(z) \]

\[ = \xi C_2 \cosh(\beta z) + A \frac{\xi^2}{\alpha^2 - \beta^2} \cosh(\alpha z) + \beta E_1 \cosh(\beta z) - \alpha \frac{\alpha}{\alpha^2 - \beta^2} \cosh(\alpha z) \]

\[ = (\xi C_2 + \beta E_1) \cosh(\beta z) + A \frac{\xi^2 - \alpha^2}{\alpha^2 - \beta^2} \cosh(\alpha z) \quad (6.30) \]

\[ = A \frac{\mu}{\lambda + \mu} \cosh(\alpha z) \]

Note that

\[ \frac{\xi^2 - \alpha^2}{\alpha^2 - \beta^2} = \frac{\xi^2 - \left( \frac{\omega^2 \rho}{\lambda + 2\mu} \right)}{\xi^2 - \left( \frac{\omega^2 \rho}{\lambda + 2\mu} \right) - \left( \frac{\omega^2 \rho}{\mu} \right)} = \frac{\mu}{\lambda + 2\mu} \]

Equation (6.30) yields \( \xi C_2 + \beta E_1 = 0 \), i.e., \( C_2 = -\frac{\beta}{\xi} E_1 \). Hence, the displacements in the axial case are
\[ u_r = J_0(\xi r)e^{i\omega t} \left[ -B \frac{\beta}{\xi} \cosh \beta z + A \frac{\xi}{\alpha^2 - \beta^2} \cosh \alpha z \right] \]
\[ u_z = J_0(\xi r)e^{i\omega t} \left[ B \sinh \beta z - A \frac{\alpha}{\alpha^2 - \beta^2} \sinh \alpha z \right] \] (6.31)

Similarly, in the flexural case,
\[ u_r = J_1(\xi r)e^{i\omega t} \left[ -B \frac{\beta}{\xi} \sinh \beta z + A \frac{\xi}{\alpha^2 - \beta^2} \sinh \alpha z \right] \]
\[ u_z = J_0(\xi r)e^{i\omega t} \left[ B \cosh \beta z - A \frac{\alpha}{\alpha^2 - \beta^2} \cosh \alpha z \right] \] (6.32)

6.2.1 Axial vibrations

Once the displacement functions are known, the strains and stress can be derived. For axial vibration, recall from Equation (6.2):
\[ \Delta = A \frac{\mu}{\lambda + \mu} \cosh \alpha z J_0(\xi r)e^{i\omega t} \] (6.33)

Substitution of Equation (6.31) into the definition of strains yields:
\[ \varepsilon_{rr} = \frac{\partial u_r}{\partial r} = \left[ -B \frac{\beta}{\xi} \cosh \beta z + A \frac{\xi}{\alpha^2 - \beta^2} \cosh \alpha z \right] \left( \frac{\xi}{r} J_0(\xi r) \right)e^{i\omega t} \]
\[ \varepsilon_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} = \frac{1}{r} J_1(\xi r)e^{i\omega t} \left[ -B \frac{\beta}{\xi} \cosh \beta z + A \frac{\xi}{\alpha^2 - \beta^2} \cosh \alpha z \right] \]
\[ \varepsilon_{zz} = \frac{\partial u_z}{\partial z} = J_0(\xi r)e^{i\omega t} \left[ B \cosh \beta z - A \frac{\alpha}{\alpha^2 - \beta^2} \cosh \alpha z \right] \]
\[ \varepsilon_{\theta r} = \frac{1}{2} \left( \frac{1}{r} \frac{\partial u_\theta}{\partial r} + \frac{\partial u_r}{\partial r} - \frac{u_\theta}{r} \right) = 0 \]
\[ \varepsilon_{r z} = \frac{1}{2} \left( \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right) = \frac{1}{2} J_1(\xi r)e^{i\omega t} \left[ -B \frac{\xi^2 + \beta^2}{\xi} \sinh \beta z + A \frac{2\alpha \xi}{\alpha^2 - \beta^2} \sinh \alpha z \right] \]
\[ \varepsilon_{\theta z} = \frac{1}{2} \left( \frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right) = 0 \] (6.34)
Using the Equation (6.33) and (6.34), the stresses are derived as:

\[
\sigma_z = \lambda \Delta + 2\mu \varepsilon_{zz}
\]
\[
= \left[ \lambda \left( A \frac{\varepsilon^2 - \alpha^2}{\alpha^2 - \beta^2} \cosh \alpha z \right) + 2\mu \left( \beta B \cosh \beta z - A \frac{\beta^2}{\alpha^2 - \beta^2} \cosh \alpha z \right) \right] J_0(\xi r) e^{j\omega t}
\]
\[
= \mu \left( 2\beta B \cosh \beta z - A \frac{2\alpha^2}{\alpha^2 - \beta^2} \cosh \alpha z + A \frac{\lambda}{\alpha^2 - \beta^2} \cosh \alpha z \right) J_0(\xi r) e^{j\omega t}
\]
\[
= \mu \left( 2\beta B \cosh \beta z - A \frac{\varepsilon^2 + \beta^2}{\alpha^2 - \beta^2} \cosh \alpha z \right) J_0(\xi r) e^{j\omega t}
\]
\[
(6.35)
\]

\[
\sigma_{rz} = 2\mu \varepsilon_{rz}
\]
\[
= \mu \left[ \frac{2\beta B}{\xi} \frac{\varepsilon^2 + \beta^2}{\alpha^2 - \beta^2} - \sinh \beta z + A \frac{2\alpha \xi}{\alpha^2 - \beta^2} \sin \alpha z \right] J_1(\xi r) e^{j\omega t}
\]
\[
(6.36)
\]

Now, impose the free-surface boundary conditions on these two stresses,

\[
\sigma_z \big|_{z=d} = \mu \left( 2\beta B \cosh \beta d - A \frac{\varepsilon^2 + \beta^2}{\alpha^2 - \beta^2} \cosh \alpha d \right) J_0(\xi r) e^{j\omega t} = 0
\]
\[
(6.37)
\]

\[
\sigma_{rz} \big|_{z=d} = \mu \left[ \frac{2\beta B}{\xi} \frac{\varepsilon^2 + \beta^2}{\alpha^2 - \beta^2} - \sinh \beta d + A \frac{2\alpha \xi}{\alpha^2 - \beta^2} \sin \alpha d \right] J_1(\xi r) e^{j\omega t} = 0
\]

For non-zero solutions, the determinant of the coefficients of A and B in Equation (6.36) must be equal to zero, i.e.,

\[
\begin{vmatrix}
2\beta \cosh \beta d & -\frac{\varepsilon^2 + \beta^2}{\alpha^2 - \beta^2} \cosh \alpha d \\
-\frac{\varepsilon^2 + \beta^2}{\xi} \sinh \beta d & \frac{2\alpha \xi}{\alpha^2 - \beta^2} \sin \alpha d
\end{vmatrix} = 0
\]

This yields the characteristic equation:

\[
\frac{\tanh \beta d}{\tanh \alpha d} = \frac{4\alpha \beta \varepsilon^2}{(\varepsilon^2 + \beta^2)^2}
\]
\[
(6.38)
\]
This is the familiar Rayleigh-Lamb frequency equation for the “axial” modes. At the same time, the ratio of $A$ and $B$ is determined as

\[
\frac{B}{A} = \frac{1}{2\beta} \frac{\xi^2 + \beta^2}{\alpha^2 - \beta^2} \cdot \frac{\cosh \alpha d}{\cosh \beta d} = \frac{2\alpha \xi^2}{(\alpha^2 - \beta^2)(\xi^2 + \beta^2)} \cdot \frac{\sinh \alpha d}{\sinh \beta d}
\]  

(6.39)

Hence that the final form of the displacements is

\[u_r = A \left( \frac{\xi}{\alpha^2 - \beta^2} \right) J_1(\xi r) \left[ -\frac{\xi^2 + \beta^2}{2\xi^2} \frac{\cosh \alpha d}{\cosh \beta d} \cosh \beta z + \cosh \alpha z \right] e^{i\omega t}\]

\[u_z = -A \left( \frac{\alpha}{\alpha^2 - \beta^2} \right) J_0(\xi r) \left[ -\frac{\xi^2}{\xi^2 + \beta^2} \frac{\sinh \alpha d}{\sinh \beta d} \sinh \beta z + \sinh \alpha z \right] e^{i\omega t}\]  

(6.40)

6.2.2 Flexural vibrations

For flexural vibrations, Equation (6.2) gives the dilatation as

\[\Delta = A \frac{\mu}{\lambda + \mu} \sinh \alpha z J_0(\xi r) e^{i\omega t}\]  

(6.41)

Substitution of Equation (6.32) into the definition of strains yields:

\[\varepsilon_{rr} = \frac{\partial u_r}{\partial r} = \left[ -B \frac{\beta}{\xi} \sinh \beta z + A \frac{\xi}{\alpha^2 - \beta^2} \sinh \alpha z \right] \left( \frac{\xi}{\rho} J_0(\xi r) - \frac{1}{r} J_1(\xi r) \right) e^{i\omega t}\]

\[\varepsilon_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} = \frac{1}{r} J_1(\xi r) e^{i\omega t} \left[ -B \frac{\beta}{\xi} \sinh \beta z + A \frac{\xi}{\alpha^2 - \beta^2} \sinh \alpha z \right]\]

\[\varepsilon_z = \frac{\partial u_z}{\partial z} = J_0(\xi r) e^{i\omega t} \left[ \beta B \sinh \beta z - A \frac{\alpha^2}{\alpha^2 - \beta^2} \sinh \alpha z \right]\]

\[\varepsilon_{r\theta} = \frac{1}{2} \left( \frac{1}{r} \frac{\partial u_\theta}{\partial r} + \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} \right) = 0\]

\[\varepsilon_{rz} = \frac{1}{2} \left( \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right) = \frac{1}{2} J_1(\xi r) e^{i\omega t} \left[ -B \frac{\xi^2 + \beta^2}{\xi} \cosh \beta z + A \frac{2\alpha \xi}{\alpha^2 - \beta^2} \cosh \alpha z \right]\]

\[\varepsilon_{\theta z} = \frac{1}{2} \left( \frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) = 0\]

(6.42)
Using Equation (6.41) and (6.42), the stresses are derived as:

\[
\sigma_z = \lambda \Delta + 2 \mu \varepsilon_{zz} \\
= \left[ \lambda \left( A \frac{x^2 - \alpha^2}{\alpha^2 - \beta^2} \sinh \alpha z \right) + 2 \mu \left( \beta B \sinh \beta z - A \frac{\alpha^2}{\alpha^2 - \beta^2} \sinh \alpha z \right) \right] J_0 (\xi r) e^{i\theta} \\
= \mu \left( 2 \beta B \sinh \beta z - A \frac{2\alpha^2}{\alpha^2 - \beta^2} \sinh \alpha z + A \frac{\lambda}{\alpha^2 - \beta^2} \sinh \alpha z \right) J_0 (\xi r) e^{i\theta} \\
= \mu \left( 2 \beta B \sinh \beta z - A \frac{\xi^2 + \beta^2}{\alpha^2 - \beta^2} \sinh \alpha z \right) J_0 (\xi r) e^{i\theta}
\] (6.43)

\[
\sigma_{rr} = 2 \mu \varepsilon_{rr} \\
= \mu \left[ -B \frac{\xi^2 + \beta^2}{\xi} \cosh \beta z + A \frac{2\alpha \xi}{\alpha^2 - \beta^2} \cosh \alpha z \right] J_1 (\xi r) e^{i\theta}
\] (6.44)

Imposing the free-surface boundary conditions at the upper and lower surfaces, set these two stresses to zero, i.e.,

\[
\sigma_z \big|_{z=d} = \mu \left( 2 \beta B \sinh \beta d - A \frac{\xi^2 + \beta^2}{\alpha^2 - \beta^2} \sinh \alpha d \right) J_0 (\xi r) e^{i\theta} = 0
\]

\[
\sigma_{rr} \big|_{z=d} = \mu \left[ -B \frac{\xi^2 + \beta^2}{\xi} \cosh \beta d + A \frac{2\alpha \xi}{\alpha^2 - \beta^2} \cosh \alpha d \right] J_1 (\xi r) e^{i\theta} = 0
\] (6.45)

Setting the determinant of coefficients of A and B in Equation (6.36) to zero,

\[
\begin{vmatrix}
2 \beta \sinh \beta d & -\frac{\xi^2 + \beta^2}{\alpha^2 - \beta^2} \sinh \alpha d \\
-\frac{\xi^2 + \beta^2}{\xi} \cosh \beta d & \frac{2\alpha \xi}{\alpha^2 - \beta^2} \cosh \alpha d
\end{vmatrix} = 0
\]

yields the characteristic equation

\[
\frac{\tanh \beta d}{\tanh \alpha d} = \frac{(\xi^2 + \beta^2)^2}{4\alpha \beta \xi^3}
\] (6.46)
This is the familiar Rayleigh-Lamb frequency equation for the “flexural” modes. At the same time the ratio of A and B is determined as:

\[
\frac{B}{A} = \frac{1}{2\beta} \frac{\xi^2 + \beta^2}{\alpha^2 - \beta^2} \frac{\sinh \alpha d}{\sinh \beta d} = \frac{2\alpha \xi^2}{(\alpha^2 - \beta^2)(\xi^2 + \beta^2)} \frac{\cosh \alpha d}{\cosh \beta d}
\]  

(6.47)

Hence, that the final form of the displacements are

\[
u_r = A \left( \frac{\xi}{\alpha^2 - \beta^2} \right) J_1(\xi \tau) \left[ \frac{\xi^2 + \beta^2}{2 \xi^2} \frac{\sinh \alpha d}{\sinh \beta d} \sinh \beta z + \sinh \alpha z \right] e^{i\omega t}
\]

\[
u_z = -A \left( \frac{\alpha}{\alpha^2 - \beta^2} \right) J_0(\xi \tau) \left[ -\frac{2 \xi^2}{\xi^2 + \beta^2} \frac{\cosh \alpha d}{\cosh \beta d} \cosh \beta z + \cosh \alpha z \right] e^{i\omega t}
\]  

(6.48)

6.2.3 The large R asymptotic behavior

Different from the straight crested Lamb waves, which follow the trigonometric functions. The circular crested Lamb waves follow the Bessel functions. But for large value of r, the Bessel functions converge to an asymptotic expression (Abramowitz, 1965)

\[
J_v(z) \to \sqrt{\frac{2}{(\pi z)}} \cos \left( \frac{z - \frac{1}{2} \nu \pi - \frac{1}{4} \pi}{2} \right)
\]  

(6.49)

In our case,

\[
J_0(\xi \tau) \mid_{\xi \tau \to \infty} \to \sqrt{\frac{2}{\pi \xi \tau}} \cos \left( \frac{\xi \tau - \pi / 4}{\sqrt{\pi \xi \tau}} \right) = \frac{\cos \xi \tau + \sin \xi \tau}{\sqrt{\pi \xi \tau}}
\]

\[
J_1(\xi \tau) \mid_{\xi \tau \to \infty} \to \sqrt{\frac{2}{\pi \xi \tau}} \cos \left( \frac{\xi \tau - 3\pi / 4}{\sqrt{\pi \xi \tau}} \right) = \frac{-\cos \xi \tau + \sin \xi \tau}{\sqrt{\pi \xi \tau}}
\]  

(6.50)

This means that for large values of \(\xi \tau\) the displacements pattern becomes periodic in distance. In this limit, the displacements reduce to the following forms.

For axial vibration:
\[
\begin{align*}
    u_r &= A \left( \frac{\xi}{\alpha^2 - \beta^2} \right) \sin \xi r - \cos \xi r \left[ \cosh \alpha z - \frac{\xi^2 + \beta^2}{2\xi^2} \frac{\cosh \alpha b}{\cosh \beta b} \right] e^{i\omega t} \\
    u_z &= -A \left( \frac{\alpha}{\alpha^2 - \beta^2} \right) \sin \xi r + \cos \xi r \left[ \sinh \alpha z - \frac{2\xi^2}{\xi^2 + \beta^2} \frac{\sinh \alpha b}{\sinh \beta b} \right] e^{i\omega t}
\end{align*}
\] (6.51)

For flexural vibration

\[
\begin{align*}
    u_r &= A \left( \frac{\xi}{\alpha^2 - \beta^2} \right) \sin \xi r - \cos \xi r \left[ \sinh \alpha z - \frac{\xi^2 + \beta^2}{2\xi^2} \frac{\sinh \alpha b}{\sinh \beta b} \right] e^{i\omega t} \\
    u_z &= -A \left( \frac{\alpha}{\alpha^2 - \beta^2} \right) \sin \xi r + \cos \xi r \left[ \cosh \alpha z - \frac{2\xi^2}{\xi^2 + \beta^2} \frac{\cosh \alpha b}{\cosh \beta b} \right] e^{i\omega t}
\end{align*}
\] (6.52)

Figure 6.2  Comparison of Bessel function and their asymptotic expression at large \( \xi_r \) value. The numbers indicate the local ratio between apparent wavelength and the asymptotic wavelength.

These two sets of equations are periodic in \( \xi_r \). The displacement patterns across the thickness are controlled by the terms in the bracket, which are of the same form as for the straight crested Lamb waves. If we define "half wave length" \( \lambda'/2 \) as the distance between adjacent zeros of the radial displacement, \( u_r \), the limiting wave length \( \lambda' \) is given by the wavelength of \( \sin(\xi r) \) and \( \cos(\xi r) \), i.e.:

\[
\lim_{r \rightarrow \infty} \lambda' = \lambda'_{\infty} = \frac{2\pi}{\xi}
\] (6.53)

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Figure 6.2 shows the comparison between $J_0 J_1$ and their asymptotic expressions. Note that the ratio between the apparent wavelength and the asymptotic wavelength, $\lambda_0'$, approaches unity after 4 oscillations.

6.3 CONCLUSION

Circular crested wave in a plate was studied by Goodman and presented in his paper. The study presented in this dissertation is aimed to demonstrate the understanding of this type of wave. Displacements of this type of wave are governed by Bessel functions, which is more complicated than the straight crested Lamb wave. Although have this distinction, the Rayleigh-Lamb frequency equation retains and still could be used to determine the phase velocity of the waves. Obviously, the displacements, strain, and stress distribution will show a different pattern. And it was noticed that beyond a small distance from the origin, the circular crested wave will be very close to the form of the straight crested wave, and the straight crested Lamb waves are the limits of the circular crested waves.

In elastic wave applications this phenomena is very useful, by using the simpler Lamb wave functions, the circular crested waves can be estimated with very small error if the distance from the origin is more than 4 to 5 nodes.

6.4 APPENDIX

6.4.1 Elastic equations

Elastic equations in cylindrical coordinate system (Graff, 1975 pp. 600)
Figure 6.3  Cylindrical coordinate system (Graff, 1975)

\[
\begin{align*}
\varepsilon_{rr} &= \frac{\partial u_r}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \frac{u_r}{r}, \quad \varepsilon_{\theta\theta} = \frac{\partial u_\theta}{\partial \theta}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} \\
\varepsilon_{r\theta} &= \frac{1}{2} \left( \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} \right) \\
\varepsilon_{rz} &= \frac{1}{2} \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) \\
\varepsilon_{\theta z} &= \frac{1}{2} \left( \frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right)
\end{align*}
\]

(6.54)

\[
\begin{align*}
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \sigma_{rr} &- \frac{S_{\theta\theta}}{r} + \rho f_r = \rho \ddot{u}_r \\
\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + 2 \sigma_{r\theta} &+ \rho f_\theta = \rho \ddot{u}_\theta \\
\frac{\partial \sigma_{r z}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + 1 \tau_{r z} &+ \rho f_z = \rho \ddot{u}_z
\end{align*}
\]

(6.55)

\[
\begin{align*}
\text{grad}(\phi) &= \frac{\partial \phi}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{e}_\theta + \frac{\partial \phi}{\partial z} \hat{e}_z \\
\text{div}(\vec{A}) &= \frac{1}{r} \frac{\partial}{\partial r} \left( r A_r \right) + \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \\
\text{curl}(\vec{A}) &= \left( \frac{1}{r} \frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} \right) \hat{e}_r + \left( \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \hat{e}_\theta + \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r A_\theta \right) - \frac{1}{r} \frac{\partial A_r}{\partial \theta} \right] \hat{e}_z
\end{align*}
\]

(6.56)

\[
\begin{align*}
\nabla^2 \phi &= \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} \\
\nabla^2 \vec{A} &= \left( \nabla^2 A_r - \frac{A_r}{r^2} \frac{2 \partial A_\theta}{\partial \theta} \right) \hat{e}_r + \left( \nabla^2 A_\theta - \frac{A_\theta}{r^2} \frac{2 \partial A_r}{\partial r} \right) \hat{e}_\theta + \nabla^2 A_z \hat{e}_z
\end{align*}
\]
6.4.2 Bessel function

Bessel functions are reviewed by Doyle, (1997, pp.)

An equation of the form

\[ z \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - n^2)w = 0, \quad n \geq 0 \]

is called Bessel's equation and has the two independent solutions \( J_n(z) \) and \( Y_n(z) \) such that:

\[ w(z) = C_1 J_n(z) + C_2 Y_n(z) \]

where \( J_n(z) \) and \( Y_n(z) \) are Bessel functions of the first and second kind, respectively.

A related Bessel equation (which can be obtained by replacing \( z \) with \( iz \)) is

\[ z \frac{d^2 w}{dz^2} + z \frac{dw}{dz} - (z^2 - n^2)w = 0, \quad n \geq 0 \]

and has the two independent solutions \( I_n(z) \) and \( K_n(z) \)

\[ w(z) = C_3 I_n(z) + C_4 K_n(z) \]

where \( I_n \), \( K_n \) are called modified Bessel functions of the first and second kind, respectively.
CHAPTER 7 ENERGY PROPAGATION; LAMB MODES ORTHOGONALITY

The propagation of elastic wave is concomitant with a flux of energy. Before the discussion of energy propagation in a plate, a few terms need to be defined.

![Diagram of energy density and power intensity](image)

**Figure 7.1** Power input to a cylinder with a cross-section of unit area and length of one wavelength. $c_e$ is the energy propagation velocity and $T$ is the period of power fluctuation.

As shown in Figure 7.1, the power intensity $\mathcal{P}$ is the power applied on a unit area; energy propagation velocity is $c_e$; and the power fluctuation is assumed to be time-harmonic with period $T$. The energy density $\mathcal{H}$ is the energy contained in a unit volume.

The average of a time-harmonic function $f = f(t)$ is

$$
\langle f \rangle = \frac{1}{T} \int_{0}^{T} f(t) dt
$$

(7.1)

where $T$ is the period of the function in time.

Then the time-averaged power input per unit area is

$$
\langle \mathcal{P} \rangle = \frac{1}{T} \int_{0}^{T} \mathcal{P} dt
$$

(7.2)
The time averaged energy density is

\[
\langle \mathcal{H} \rangle = \frac{1}{T} \int_0^T \mathcal{H} dt
\]  

(7.3)

If assume the medium start at rest (quiescent past), then we will have \( \langle \mathcal{P} \rangle T = \langle \mathcal{H} \rangle c_s T \).

Hence, \( \langle \mathcal{P} \rangle = \langle \mathcal{H} \rangle c_s \), and the energy propagation velocity can be calculated as:

\[
c_e = \frac{\langle \mathcal{P} \rangle}{\langle \mathcal{H} \rangle} \tag{7.4}
\]

When the function is complex, only the real part is of concern. A frequently used quantity is the averaged product of the real part of two complex functions, which is represented by \( \langle \text{Re}(f) \times \text{Re}(g) \rangle \). If the two function are time-harmonic, i.e., \( f = Fe^{i\omega t} \), and \( g = Ge^{i(\omega t - \phi)} \), where \( F \) and \( G \) are real functions, then the following relation holds:

\[
\langle \text{Re}(f) \times \text{Re}(g) \rangle = \frac{1}{2} \text{Re}(f \cdot \overline{g}) \tag{7.5}
\]

where \( \overline{g} \) is the complex conjugate of \( g \). This can be proved as following:

\[
\text{Re}(f) \times \text{Re}(g) = FG \cos(\omega t) \cos(\omega t - \phi) = \frac{1}{2} FG \left[ \cos(\phi) + \cos(2\omega t - \phi) \right]
\]

\[
\langle \text{Re}(f) \times \text{Re}(g) \rangle = \frac{1}{2} FG \frac{1}{T} \left[ \int_0^T \cos(\phi) dt + \int_0^T \cos(2\omega t - \phi) dt \right]
\]

\[
= \frac{1}{2} FG \frac{1}{T} \left[ T \cos(\phi) + 0 \right] = \frac{1}{2} FG \cos(\phi)
\]

\[
\frac{1}{2} \text{Re}(f \cdot \overline{g}) = \frac{1}{2} \text{Re}\left[ FGe^{i\omega t} e^{-i(\omega t + \phi)} \right] = \frac{1}{2} \text{Re}(FGe^{-i\phi}) = \frac{1}{2} FG \cos(\phi)
\]

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\[ : (\text{Re}(f) \times \text{Re}(g)) = \frac{1}{2} \text{Re}(f \cdot \overline{g}) \]

### 7.1 Energy Propagation of Pressure Wave in Plates

![Diagram of a plate with pressure wave](image)

Figure 7.2 Pressure wave in a plate with thickness 2d. The stress is uniformly distributed in the thickness.

A pressure wave in plate is represented by

\[ u_x = Ae^{i(kx-\omega t)} \quad (7.6) \]

where particle displacement is uniformly distributed in the thickness. The strain is calculated as

\[ \varepsilon_{xx} = \frac{du_x}{dx} = ike^{i(kx-\omega t)} \quad (7.7) \]

while the stress is calculated as

\[ \sigma_{xx} = -E\varepsilon_{xx} = -E \frac{du_x}{dx} = -iEke^{i(kx-\omega t)} \quad (7.8) \]

and the particle velocity is

\[ \dot{u}_x = -i\omega Ae^{i(kx-\omega t)} \quad (7.9) \]

The power intensity is the product of the stress and the particle velocity, which is
\[ \mathcal{P} = \sigma_x \dot{u}_x \quad (7.10) \]

Then the averaged power intensity input to the plate is

\[ \frac{1}{2d} \int_{-d}^{d} \mathcal{P} dy = \frac{1}{2d} \sigma_x \dot{u}_x y \bigg|_{-d}^{d} = \sigma_x \dot{u}_x \quad (7.11) \]

Using the definition of Equation (7.2), and the relation in Equation (7.5), the time averaged power intensity is

\[
\langle \mathcal{P} \rangle = \langle \text{Re} (\sigma_x) \text{Re} (\dot{u}_x) \rangle = \frac{1}{2} \text{Re} (\sigma_x \bar{\dot{u}}_x) = \frac{1}{2} \text{Re} \left[ \left( -iE A e^{i(kz - \omega t)} \right) \left( i\omega A e^{-i(kz - \omega t)} \right) \right] = \frac{1}{2} E_k \omega A^2 \quad (7.12)
\]

Recall that wave number \( k = \omega / c \), where \( c = \sqrt{E / \rho} \) is the phase velocity. Hence

\[
\langle \mathcal{P} \rangle = \frac{1}{2} E \omega^2 A^2 \frac{1}{c} = \frac{1}{2} \left( \frac{c^2 \rho}{\omega^2} \right) \omega^2 A^2 \frac{1}{c} = \frac{1}{2} A^2 \omega^2 \rho c \quad (7.13)
\]

The energy density, \( \mathcal{H} \), consists of kinetic energy density \( \mathcal{K} \) and strain energy density \( \mathcal{U} \).

The kinetic energy density is defined as

\[ \mathcal{K} = \frac{1}{2} \rho \dot{u}_x^2 \quad (7.14) \]

The time averaged kinetic energy density is

\[
\langle \mathcal{K} \rangle = \frac{1}{2} \rho \langle \left[ \text{Re} (\dot{u}_x) \right]^2 \rangle = \frac{1}{4} \rho \text{Re} \left( \dot{u}_x \bar{\dot{u}}_x \right) = \frac{1}{4} A^2 \omega^2 \rho \quad (7.15)
\]

where \( \bar{u}_x = i\omega A e^{-i(kz - \omega t)} \), and Equation (7.5) was applied.

The strain energy density is defined as
\[ U = \frac{1}{2} \sigma_{xx} \varepsilon_{xx} \]  

(7.16)

The time averaged strain energy density is

\[
\langle U \rangle = \frac{1}{2} \langle \text{Re} (\sigma_{xx}) \text{Re} (\varepsilon_{xx}) \rangle = \frac{1}{4} \text{Re} (\sigma_{xx} \bar{\varepsilon}_{xx}) \\
= \frac{1}{4} E A^2 k^2 = \frac{1}{4} E A^2 \frac{\omega^2}{c^2} = \frac{1}{4} E A^2 \frac{\omega^2}{E/\rho} = \frac{1}{4} A^2 \omega^2 \rho
\]

(7.17)

where \( \bar{\varepsilon}_{xx} = i k A e^{-i(kx-\omega t)} \), and Equation (7.5) was applied.

From Equations (7.15) and (7.17), it can be noticed that \( \langle \mathcal{K} \rangle = \langle \mathcal{U} \rangle \). Hence, for time-harmonic waves, the time averaged energy density \( \langle \mathcal{H} \rangle \) is equally distributed between the time averages of the kinetic energy density \( \langle \mathcal{K} \rangle \) and strain energy density \( \langle \mathcal{U} \rangle \), and we have

\[
\langle \mathcal{H} \rangle = \langle \mathcal{K} \rangle + \langle \mathcal{U} \rangle = \frac{1}{2} A^2 \omega^2 \rho
\]

(7.18)

Substituting Equations (7.13) and (7.18) into Equation (7.4) yields the energy propagation velocity

\[
c_e = \frac{\langle \mathcal{P} \rangle}{\langle \mathcal{H} \rangle} = \frac{1}{2} \frac{A^2 \omega^2 \rho c}{A^2 \omega^2 \rho} = c
\]

(7.19)

Hence, it can be concluded that for this simple non-dispersive wave, the energy propagation speed is the same as the phase velocity.
7.2 Energy Propagation of SH (Shear Horizontal) Wave in Plates

The governing equation for SH wave is

$$\nabla^2 u_z = \frac{1}{c_s^2} \frac{\partial^2 u_z}{\partial t^2}$$  \hspace{1cm} (7.20)

The displacement $u_z$ is assumed as:

$$u_z = f(y) e^{i(kx - \omega t)}$$  \hspace{1cm} (7.21)

where

$$f(y) = B_1 \sin q_n y + B_2 \cos q_n y$$  \hspace{1cm} (7.22)

Application of stress-free boundary conditions on the upper and lower plate surfaces yields:

$$B_1 = 0, \quad q_n d = 2n \frac{\pi}{2} \quad \text{for symmetric modes}$$

$$B_2 = 0, \quad q_n d = (2n + 1) \frac{\pi}{2} \quad \text{for antisymmetric modes}$$

$$q_n^2 = \frac{\omega^2}{c_s^2} - k^2$$

where $n = 0, 1, 2, \ldots$ are mode numbers.
The shear stress is calculated as

\[ \sigma_{xz} = \mu \frac{\partial u_z}{\partial x} = i \mu k u_z \]  

(7.23)

and the particle velocity is

\[ \dot{u}_z = -i \omega u_z \]  

(7.24)

The power intensity is the product of the stress and the velocity, i.e.,

\[ \Phi = \sigma_{xz} \dot{u}_z \]  

(7.25)

Hence the power input intensity is

\[ \int_{-d}^{d} \Phi dy = \int_{-d}^{d} \sigma_{xz} \dot{u}_z dy \]  

(7.26)

Using the definition of Equation (7.2), and the relation of Equation (7.5), the time averaged power intensity is

\[ \langle \Phi \rangle = \left( \int_{-d}^{d} \sigma_{xz} \dot{u}_z dy \right) = \frac{1}{T} \int_{-d}^{d} \sigma_{xz} \dot{u}_z dy dt \]

\[ = \int_{-d}^{d} \left( \frac{1}{T} \int_{0}^{T} \sigma_{xz} \dot{u}_z dt \right) dy = \int_{-d}^{d} \langle \sigma_{xz} \dot{u}_z \rangle dy \]

(7.27)

\[ = \frac{1}{2} \text{Re} \int_{-d}^{d} \sigma_{xz} \dot{u}_z dy \]

\[ = \frac{1}{2} \text{Re} \int_{-d}^{d} (i \mu k u_z) (i \omega u_z) dy = -\frac{1}{2} \mu k \omega \text{Re} \int_{-d}^{d} f^2 (y) dy \]

The energy density \( \mathcal{H} \) consists of kinetic energy density \( \mathcal{K} \) and strain energy density \( \mathcal{U} \).

The kinetic energy density is defined by

\[ \mathcal{K} = \frac{1}{2} \rho \dot{u}_z^2 \]  

(7.28)
The time averaged kinetic energy density is

\[
\left\langle \int_{-d}^{d} \mathcal{K} dy \right\rangle = \frac{1}{2} \rho \left\langle \left[ \text{Re}(\ddot{u}_x) \right]^2 \right\rangle dy = \frac{1}{4} \rho \text{Re} \int_{-d}^{d} (\ddot{u}_x \ddot{u}_x) dy \\
= \frac{1}{4} \rho \text{Re} \int_{-d}^{d} [(\ddot{u}_x)]^2 dy = \frac{1}{4} \rho \omega^2 \int_{-d}^{d} f^2(y) dy
\]

(7.29)

where \( \ddot{u}_x = i \omega f(y) e^{-i(kx-\omega t)} \), and Equation (7.5) was applied.

The strain energy density is defined by

\[
\mathcal{U} = \frac{1}{2} \sigma_{xx} \varepsilon_{xx}
\]

(7.30)

The time averaged strain energy density is taken equal to the time averaged kinetic energy density, and hence:

\[
\left\langle \mathcal{K} \right\rangle + \left\langle \mathcal{U} \right\rangle = 2 \left\langle \mathcal{K} \right\rangle = \frac{1}{2} \rho \omega^2 \int_{-d}^{d} f^2(y) dy
\]

(7.31)

As an example, take the anti-symmetric \( A_0 \) mode

\[
f(y) = B_i \sin \frac{\pi}{2d} \frac{y}{d}
\]

(7.32)

Substituting Equation (7.32) into Equation (7.27) yields

\[
\left\langle \mathcal{Q} \right\rangle = -\frac{1}{2} \mu k \omega B_i^2 d
\]

(7.33)

where the following integral was used

\[
\int_{-d}^{d} f^2(y) dy = B_i^2 \int_{-d}^{d} \sin^2 \left( \frac{\pi}{2d} \frac{y}{d} \right) dy = B_i^2 \frac{1}{2} \int_{-d}^{d} 1 - \cos \left( \frac{\pi}{d} \frac{y}{d} \right) dy = B_i^2 d
\]

(7.34)

Substituting Equation (7.32) into Equation (7.31) gives
\[
\langle y \rangle = \frac{1}{2} \rho \omega^2 \int_{-d}^{d} f^2(y) dy = \frac{1}{2} \rho \omega^2 B_i^2 d
\]  
(7.35)

Substituting Equations (7.33) and (7.35) into Equation (7.4), the energy propagation velocity in a SH wave is

\[
c_e = \frac{\langle \Phi \rangle}{\langle y \rangle} = \frac{\frac{1}{2} \mu k \omega B_i^2 d}{\frac{1}{2} \rho \omega^2 B_i^2 d} = \frac{\mu k}{\rho \omega}
\]
(7.36)

Recall that the wave number \( k = \omega / c \), and the shear wave phase velocity \( c_s = \sqrt{\mu / \rho} \).

Hence, Equation (36) becomes:

\[
c_e = \frac{\mu k}{\rho \omega} = \frac{c_s^2}{c}
\]
(7.37)

Using Equation (22):

\[
q_n^2 = \frac{\omega^2}{c_s^2} - k^2 \Rightarrow \omega^2 = c_s^2 \left( q_n^2 + k^2 \right)
\]
(7.38)

By taking the derivative:

\[
2\omega \frac{d\omega}{dk} = 2c_s^2 k \Rightarrow \frac{d\omega}{dk} = \frac{c_s^2 k}{\omega} = \frac{c_s^2}{c}
\]
(7.39)

By definition \( c_g = \frac{d\omega}{dk} \), hence \( c_g = \frac{c_s^2}{c} \), which is exactly \( c_e \).

Thus, we have shown that the energy velocity has the same formula as the group velocity, which represented the speed a “group” of wave travels. Indeed, the “group” of waves actually carries the energy. So we can find out the physical meaning of the equation. For SH wave, the energy propagation speed is the same as the group velocity.
From the definition, for non-dispersive pressure waves, like the pressure wave and the $S_0$ mode of SH wave, group velocity is the same as its phase velocity. So we can conclude that the energy propagation speed is the same as the group velocity.

7.3 Orthomomality of Lamb Mode

The orthonomality of Lamb mode includes two steps:

1. Orthogonality

2. Normalization

The approach will utilize the complex reciprocity relation (Auld, 1973). First, the complex reciprocity relation will be derived from first principles. Then the Lamb modes orthogonality will be proved. Finally, the scaling factors for each Lamb mode will be determined, and normalize the Lamb modes with this factor.

7.3.1 Orthogonality

Consider the governing equation for a homogeneous isotropic elastic solid plate, following Graff (1975, pp 432)

$$\frac{\partial}{\partial x_j} \sigma_{ij} = \rho \frac{d^2}{dt^2} u_i$$

(7.40)

$$\sigma_{ij} = c_{ijkl} \frac{\partial u_k}{\partial x_l}$$

where $\sigma_{ij}$ is the stress tensor, $u$ is the displacement, $c$ is the stiffness tensor and $\rho$ is the density. The boundary condition is given by

$$n_j = \sigma_{ij} n_j = 0 \text{ at upper and lower surfaces}$$

(7.41)

where $n_j$ is the unit normal vector on the surface of the plate.
Under plane strain condition and harmonic excitation, the solution is given by

$$u(x, y, t) = u(y) e^{(\xi x - \omega t)}$$  \hspace{1cm} (7.42)

where $u(y)$ is the displacement variation in the thickness direction of the plate, $\xi$ is the wave number, $\omega$ is the circular frequency. The function $u(y)$ is now to be solved by normal mode expansion.

Consider a volume $\delta V$ with surface area $\delta S$, and the mass density is $\rho$. It is under body force $F \delta V$ and traction force $T$ on the surface. The integrated surface force is

$$\int_{\delta S} T \cdot \hat{n} dS$$

where $\hat{n}$ is the unit vector normal to $\delta S$. Newton’s law gives

$$\int_{\delta S} T \cdot \hat{n} dS + \int_{\delta V} F dV = \int_{\delta V} \frac{\partial^2 u}{\partial t^2} dV$$  \hspace{1cm} (7.43)

where $u$ is the displacement vector. For an infinitesimal volume, Equation (7.43) can be written as

$$\int_{\delta S} T \cdot \hat{n} dS \bigg|_{\delta V} = \rho \frac{\partial^2 u}{\partial t^2} - F$$  \hspace{1cm} (7.44)

and the limit of left-hand side of Equation (7.44) as $\delta V \to 0$ is the divergence of the stress and can be written as

$$\nabla \cdot T = \lim_{\delta V \to 0} \frac{\int_{\delta S} T \cdot \hat{n} dS}{\delta V}$$  \hspace{1cm} (7.45)

Hence Equation (7.44) becomes
\[ \nabla \cdot \mathbf{T} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \mathbf{F} \] (7.46)

This is the equation of motion.

The strain is given by

\[ \mathbf{S} = \frac{1}{2} \left( \nabla \mathbf{u} + \nabla^T \mathbf{u} \right) = \nabla_s \mathbf{u} \] (7.47)

where \( \nabla^T \mathbf{u} \) is the conjugate of \( \nabla \mathbf{u} \), and

\[
\nabla_s \rightarrow \begin{bmatrix}
\frac{\partial}{\partial x_1} & 0 & 0 \\
0 & \frac{\partial}{\partial x_2} & 0 \\
0 & 0 & \frac{\partial}{\partial x_3} \\
0 & \frac{\partial}{\partial x_3} & 0 \\
\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0
\end{bmatrix}
\]

Thus, the stress-strain relation is given by Hook’s law

\[ \mathbf{T} = \mathbf{c} : \mathbf{S} \]

and

\[ \mathbf{S} = \mathbf{s} : \mathbf{T} \]

where \( \mathbf{c} \) is the stiffness tensor, \( \mathbf{s} \) is the compliance tensor, and the double dot product is the summation over pairs of repeated subscripts. i.e.
\[ T_i = c_i S_j, \quad i, j = 1, 2, 3, \ldots, 6 \]

Define the particle velocity

\[ \mathbf{v} = \frac{\partial \mathbf{u}}{\partial t} \]

From Equation (7.47)

\[ \nabla_s \mathbf{v} = \nabla_s \frac{\partial \mathbf{u}}{\partial t} = \frac{\partial \nabla_s \mathbf{u}}{\partial t} = \frac{\partial \mathbf{S}}{\partial t} \tag{7.48} \]

and Equation (7.46) can be written as

\[ \nabla \cdot \mathbf{T} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \mathbf{F} = \rho \frac{\partial \mathbf{v}}{\partial t} - \mathbf{F} \tag{7.49} \]

Write Equations (7.48) and (7.49) in matrix form

\[
\begin{bmatrix}
0 & \nabla \cdot \\
\nabla, & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{v} \\
\mathbf{T}
\end{bmatrix}
= 
\begin{bmatrix}
\rho & 0 \\
0 & \mathbf{s} : 
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \mathbf{v}}{\partial t} \\
\frac{\partial \mathbf{T}}{\partial t}
\end{bmatrix}
+ 
\begin{bmatrix}
-\mathbf{F} \\
0
\end{bmatrix} 
\tag{7.50}
\]

Take the fields to be time harmonic, i.e. \( \mathbf{v}(\mathbf{r}, t) = \mathbf{v}(\mathbf{r}) e^{i\omega t} \), such that the time derivative is always

\[ \frac{\partial}{\partial t} \left( \cdot \right) = i \omega \left( \cdot \right) \]

Hence, Equation (7.50) can be written as

\[
\begin{bmatrix}
0 & \nabla \cdot \\
\nabla, & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{v} \\
\mathbf{T}
\end{bmatrix}
= 
\begin{bmatrix}
\rho & 0 \\
0 & \mathbf{s} : 
\end{bmatrix}
\begin{bmatrix}
\mathbf{v} \\
\mathbf{T}
\end{bmatrix}
+ 
\begin{bmatrix}
-\mathbf{F} \\
0
\end{bmatrix} 
\tag{7.51}
\]

Assume solution “1”, \( \mathbf{v}_1, \mathbf{T}_1, \mathbf{F}_1 \), and solutions “2”, \( \mathbf{v}_2, \mathbf{T}_2, \mathbf{F}_2 \) are two sets of solutions to Equation (7.51), the following procedures are applied to Equation (7.51):
plug $\mathbf{v}_1, \mathbf{T}_1, \mathbf{F}_1$ into Equation (7.51)

$$\begin{bmatrix} 0 & \nabla \cdot \\ \nabla \times & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{T}_1 \end{bmatrix} = i\omega \begin{bmatrix} \rho \\ 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{T}_1 \end{bmatrix} \begin{bmatrix} \mathbf{F}_1 \\ 0 \end{bmatrix}$$

(7.52)

take scalar product of Equation (7.52) and $[-\mathbf{v}_2, \mathbf{T}_2]$

$$\begin{bmatrix} -\mathbf{v}_2 & \mathbf{T}_2 \end{bmatrix} \begin{bmatrix} 0 \\ \nabla \cdot \\ \nabla \times & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{T}_1 \end{bmatrix} = i\omega \begin{bmatrix} -\mathbf{v}_2 & \mathbf{T}_2 \end{bmatrix} \begin{bmatrix} \rho \\ 0 \\ s \cdot \mathbf{T}_1 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{T}_1 \end{bmatrix} + \begin{bmatrix} -\mathbf{v}_2 \\ \mathbf{T}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_1 \\ 0 \end{bmatrix}$$

(7.53)

Interchange subscript 1 and 2 in Equation (7.53) and the equation should still valid

$$\begin{bmatrix} -\mathbf{v}_1 & \mathbf{T}_1 \end{bmatrix} \begin{bmatrix} 0 \\ \nabla \cdot \\ \nabla \times & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_2 \\ \mathbf{T}_2 \end{bmatrix} = i\omega \begin{bmatrix} -\mathbf{v}_1 & \mathbf{T}_1 \end{bmatrix} \begin{bmatrix} \rho \\ 0 \\ s \cdot \mathbf{T}_2 \end{bmatrix} \begin{bmatrix} \mathbf{v}_2 \\ \mathbf{T}_2 \end{bmatrix} + \begin{bmatrix} -\mathbf{v}_1 & \mathbf{T}_1 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 \\ 0 \end{bmatrix}$$

(7.54)

Subtract Equation (7.54) from (7.53)

$$(-\mathbf{v}_2 \cdot \nabla \cdot \mathbf{T}_1 + \mathbf{T}_2 \cdot \nabla \times \mathbf{v}_1) - (-\mathbf{v}_1 \cdot \nabla \cdot \mathbf{T}_2 + \mathbf{T}_1 \cdot \nabla \times \mathbf{v}_2) =$$

$$i\omega (-\mathbf{v}_2 \cdot \rho \cdot \mathbf{v}_1 + \mathbf{T}_2 \cdot s \cdot \mathbf{T}_1) + \mathbf{v}_2 \cdot \mathbf{F}_1 - i\omega (-\mathbf{v}_1 \cdot \rho \cdot \mathbf{v}_2 + \mathbf{T}_1 \cdot s \cdot \mathbf{T}_2) - \mathbf{v}_1 \cdot \mathbf{F}_2$$

(7.55)

Simplify Equation (7.55) by using the identity

$$\nabla \cdot (\mathbf{v} \cdot \mathbf{T}) = \mathbf{v} \cdot (\nabla \cdot \mathbf{T}) + \mathbf{T} \cdot \nabla \mathbf{v}$$

\nabla \cdot (\mathbf{v}_1 \cdot \mathbf{T}_2 - \mathbf{v}_2 \cdot \mathbf{T}_1) = i\omega (\mathbf{v}_1 \cdot \rho \cdot \mathbf{v}_2 - \mathbf{v}_2 \cdot \rho \cdot \mathbf{v}_1 + \mathbf{T}_2 \cdot s \cdot \mathbf{T}_1 - \mathbf{T}_1 \cdot s \cdot \mathbf{T}_2) + \mathbf{v}_2 \cdot \mathbf{F}_1 - \mathbf{v}_1 \cdot \mathbf{F}_2$$

(7.56)

it is easy to see that

$$\mathbf{v}_1 \cdot \rho \cdot \mathbf{v}_2 - \mathbf{v}_2 \cdot \rho \cdot \mathbf{v}_1 = 0$$

and

$$\nabla \cdot (\mathbf{v} \cdot \mathbf{T}) = \frac{\partial}{\partial x_j} (\mathbf{v}_i \cdot T_{ij}) = \mathbf{v}_i \left( \frac{\partial}{\partial x_j} T_{ij} \right) + \left( \frac{\partial}{\partial x_j} \mathbf{v}_i \right) \cdot T_{ij} = \mathbf{v} \cdot (\nabla \cdot \mathbf{T}) + \nabla \mathbf{v} \cdot \mathbf{T}$$

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\[ T_s \cdot s : T_i - T_i \cdot s : T_s = \sum_i \sum_j T^{i}_s T^{j}_s - \sum_i \sum_j T^{i}_s T^{j}_s = 0 \]

Hence, Equation (7.56) is simplified to

\[ \nabla \cdot (v_1 \cdot T - v_2 \cdot T) = v_2 \cdot F - v_1 \cdot F \quad (7.57) \]

This is the real reciprocity for acoustic field, i.e. the frequency \( \omega \) is assumed to be real constant.

A more general case will be that the frequency \( \omega \) is arbitrary, so the solution can even be non-periodic function of time.

As in the previous case, Equation (7.50) is valid

\[
\begin{bmatrix}
0 & \nabla \cdot [v] \\
\nabla_s & 0
\end{bmatrix}
\begin{bmatrix}
T
\end{bmatrix}
=\begin{bmatrix}
\rho & 0 \\
0 & s : \frac{\partial}{\partial t} [T]
\end{bmatrix}
\begin{bmatrix}
v
\end{bmatrix}
+\begin{bmatrix}
-F
\end{bmatrix}
\]

Assume solution “1”, \( v_1, T_1, F_1 \), and solutions “2”, \( v_2, T_2, F_2 \) are two sets of solutions to Equation (7.50), the following procedures are applied to Equation (7.50):

plug \( v_1, T_1, F_1 \) into Equation (7.50)

\[
\begin{bmatrix}
0 & \nabla \cdot [v_1] \\
\nabla_s & 0
\end{bmatrix}
\begin{bmatrix}
T_1
\end{bmatrix}
=\begin{bmatrix}
\rho & 0 \\
0 & s : \frac{\partial}{\partial t} [T_1]
\end{bmatrix}
\begin{bmatrix}
v_1
\end{bmatrix}
+\begin{bmatrix}
-F_1
\end{bmatrix}
\quad (7.58)
\]

take scalar product of Equation (7.58) and \( \begin{bmatrix} \tilde{v}_2 & \tilde{T}_2 \end{bmatrix} \), where \( \begin{bmatrix} \tilde{v}_2 & \tilde{T}_2 \end{bmatrix} \) is the complex conjugate of solution “2”

\[
\begin{bmatrix}
\tilde{v}_2 & \tilde{T}_2 \\
\nabla_s & 0
\end{bmatrix}
\begin{bmatrix}
0 & \nabla \cdot [v_1] \\
\nabla_s & 0
\end{bmatrix}
\begin{bmatrix}
T_1
\end{bmatrix}
=\begin{bmatrix}
\tilde{v}_2 & \tilde{T}_2
\end{bmatrix}
\begin{bmatrix}
\rho & 0 \\
0 & s : \frac{\partial}{\partial t} [T_1]
\end{bmatrix}
\begin{bmatrix}
v_1
\end{bmatrix}
+\begin{bmatrix}
\tilde{v}_2 & \tilde{T}_2
\end{bmatrix}
\begin{bmatrix}
-F_1
\end{bmatrix}
\quad (7.59)
\]

Because complex root always appear in pairs, plug \( \tilde{v}_2, \tilde{T}_2, \tilde{F}_2 \) into Equation (7.50)
\[
\begin{bmatrix}
0 & \nabla \cdot \\
\nabla_z & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{v}_2 \\
\mathbf{T}_2
\end{bmatrix}
= \begin{bmatrix}
\rho & 0 \\
0 & s \cdot \frac{\partial}{\partial t}
\end{bmatrix}
\begin{bmatrix}
\mathbf{v}_2 \\
\mathbf{T}_2
\end{bmatrix}
+ \begin{bmatrix}
-\mathbf{F}_2 \\
0
\end{bmatrix}
\] (7.60)

take scalar product of Equation (7.60) and \([\mathbf{v}_1 \ T_1]\)

\[
\begin{bmatrix}
\mathbf{v}_1 & T_1
\end{bmatrix}
\begin{bmatrix}
0 & \nabla \cdot \\
\nabla_z & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{v}_2 \\
\mathbf{T}_2
\end{bmatrix}
= \begin{bmatrix}
\mathbf{v}_1 & T_1
\end{bmatrix}
\begin{bmatrix}
\rho & 0 \\
0 & s \cdot \frac{\partial}{\partial t}
\end{bmatrix}
\begin{bmatrix}
\mathbf{v}_2 \\
\mathbf{T}_2
\end{bmatrix}
+ \begin{bmatrix}
\mathbf{v}_1 & T_1
\end{bmatrix}
\begin{bmatrix}
-\mathbf{F}_2 \\
0
\end{bmatrix}
\] (7.61)

Add Equations (7.61) and (7.59)

\[
(\mathbf{v}_2 \cdot \nabla \cdot T_1 + \mathbf{T}_2 : \nabla_s v_1) + (\mathbf{v}_1 \cdot \nabla \cdot \mathbf{T}_2 + \mathbf{T}_1 : \nabla_s \mathbf{v}_2) = \\
(\mathbf{v}_2 \cdot \rho \cdot \frac{\partial}{\partial t} \mathbf{v}_1 + \mathbf{T}_2 \cdot s \cdot \frac{\partial}{\partial t} T_1) - \mathbf{v}_2 \cdot \mathbf{F}_1 + (\mathbf{v}_1 \cdot \rho \cdot \frac{\partial}{\partial t} \mathbf{v}_2 + \mathbf{T}_1 \cdot s \cdot \frac{\partial}{\partial t} \mathbf{T}_2) - \mathbf{v}_1 \cdot \mathbf{F}_2
\] (7.62)

Simplify Equation (7.62) by using the identity

\[
\nabla \cdot (\mathbf{v} \cdot T) = \mathbf{v} \cdot (\nabla \cdot T) + T \cdot \nabla \mathbf{v}
\]

\[
\nabla \cdot (\mathbf{v}_1 \cdot T_2 + \mathbf{v}_2 \cdot T_1) = \frac{\partial}{\partial t} (\mathbf{v}_2 \cdot \rho \cdot \mathbf{v}_1 + \mathbf{T}_2 \cdot s \cdot T_1) - (\mathbf{v}_1 \cdot \mathbf{F}_2 + \mathbf{v}_2 \cdot \mathbf{F}_1)
\] (7.63)

Follow the work by Auld (1973), the proof of Lamb modes orthogonality is provided in the following paragraphs.

Assume \(\mathbf{v}_1\), \(\sigma_1\), and \(\mathbf{v}_2\), \(\sigma_2\) are two sets of time harmonic solutions to the governing equation in Equation (7.40), e.g. two Lamb modes. Where \(\mathbf{v}\) is the particle velocity field, \(\sigma\) is the stress field. They can be written as the following functions in terms of \(x\), \(y\) and \(t\), (under plane strain conditions)

Assume the velocity and stress fields of two Lamb modes are as following

\[
\mathbf{v}_1 = \mathbf{v}_m(y) e^{-i\omega x}, \quad \sigma_1 = \sigma_m(y) e^{-i\omega x}
\]
\[
\mathbf{v}_2 = \mathbf{v}_n(y) e^{-i\omega x}, \quad \sigma_2 = \sigma_n(y) e^{-i\omega x}
\] (7.64)
when substituting Equation (7.64) into (7.63), the time derivative term goes to zero

\[
\frac{\partial}{\partial t} \left( \mathbf{v}_2 \cdot \rho \cdot \mathbf{v}_1 + \mathbf{\sigma}_2 \cdot \mathbf{s} : \mathbf{\sigma}_1 \right) = \frac{\partial}{\partial t} \left( \rho \cdot \sum_i \mathbf{\tilde{v}}_i \cdot \mathbf{v}_1 + \sum_i \sum_j \tilde{\mathbf{s}}^j_i (y) \sigma^j_i (y) \right) \mathbf{e}^{i(\xi_n - \xi_s)x} \mathbf{e}^{-i\lambda t} = 0
\]

Since the “free” vibration modes are considered, the forces applied are zero, i.e. \( F_1, F_2 = 0 \).

Hence, from the complex reciprocity relation it can be derived that

\[
\nabla \cdot \left( \mathbf{\tilde{v}}_m (y) + \mathbf{v}_n (y) \right) = 0
\]  \hspace{1cm} (7.65)

Substituting Equation (7.64) into Equation (7.65) and collect the exponential terms,

\[
\left( \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} \right) \left( \mathbf{\tilde{v}}_m (y) \cdot \mathbf{\sigma}_n (y) e^{i(\xi_n - \xi_s)x} + \mathbf{v}_n (y) \cdot \tilde{\mathbf{\sigma}}_m (y) e^{i(\xi_n - \xi_s)x} \right) = 0
\]  \hspace{1cm} (7.66)

rearrange this equation

\[
i \left( \xi_n - \xi_s \right) \left( \mathbf{\tilde{v}}_m (y) \cdot \mathbf{\sigma}_n (y) + \mathbf{v}_n (y) \cdot \tilde{\mathbf{\sigma}}_m (y) \right) e^{i(\xi_n - \xi_s)x} \hat{x} = -\frac{\partial}{\partial y} \left( \mathbf{\tilde{v}}_m (y) \cdot \mathbf{\sigma}_n (y) + \mathbf{v}_n (y) \cdot \tilde{\mathbf{\sigma}}_m (y) \right) e^{i(\xi_n - \xi_s)x} \hat{y}
\]  \hspace{1cm} (7.67)

By definition

\[
\mathbf{v} = \begin{bmatrix} v_x & v_y & v_z \end{bmatrix}
\]

\[
\mathbf{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\
\sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\
\sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}
\]  \hspace{1cm} (7.68)

under plane strain condition

\[
v_z = \sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0
\]

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Hence, (7.68) can be simplified as

\[ \mathbf{v} = \begin{bmatrix} v_x \hat{x} & v_y \hat{y} \end{bmatrix} \]
\[ \sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \]  

(7.69)

Then, the identity \( \mathbf{v} \cdot \sigma \) is expanded as

\[ \mathbf{v} \cdot \sigma = \begin{bmatrix} v_x \hat{x} & v_y \hat{y} \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} = (v_x \sigma_{xx} + v_y \sigma_{yx}) \hat{x} + (v_x \sigma_{xy} + v_y \sigma_{yy}) \hat{y} \]

(7.70)

substituting this identity in Equation (7.67)

\[ i(\xi_m - \xi_r) \left[ \left( \ddot{\nu}^m_{xx} \sigma^n_{xx} + \ddot{\nu}^m_{yy} \sigma^n_{yy} \right) + \left( \nu^n_x \ddot{\sigma}^m_{xx} + \nu^n_y \ddot{\sigma}^m_{yy} \right) \right] e^{i(\xi_m - \xi_r)x} \]
\[ = - \frac{\partial}{\partial y} \left[ \left( \ddot{\nu}^m_{xx} \sigma^n_{xy} + \ddot{\nu}^m_{yy} \sigma^n_{yy} \right) + \left( \nu^n_x \ddot{\sigma}^m_{xx} + \nu^n_y \ddot{\sigma}^m_{yy} \right) \right] e^{i(\xi_m - \xi_r)x} \]  

(7.71)

Simplify Equation (7.71) by eliminating the exponential term on both sides, then take integral w.r.t. \( y \) over the thickness of the plate \([-d,+d]\)

\[ i(\xi_m - \xi_r) \int_{-d}^{d} \left[ \left( \ddot{\nu}^m_{xx} \sigma^n_{xx} + \ddot{\nu}^m_{yy} \sigma^n_{yy} \right) + \left( \nu^n_x \ddot{\sigma}^m_{xx} + \nu^n_y \ddot{\sigma}^m_{yy} \right) \right] dy \]
\[ = - \int_{-d}^{d} \frac{\partial}{\partial y} \left[ \left( \ddot{\nu}^m_{xx} \sigma^n_{xy} + \ddot{\nu}^m_{yy} \sigma^n_{yy} \right) + \left( \nu^n_x \ddot{\sigma}^m_{xx} + \nu^n_y \ddot{\sigma}^m_{yy} \right) \right] dy \]  

(7.72)

The RHS is simply the value of \( \left[ \left( \ddot{\nu}^m_{xx} \sigma^n_{xy} + \ddot{\nu}^m_{yy} \sigma^n_{yy} \right) + \left( \nu^n_x \ddot{\sigma}^m_{xx} + \nu^n_y \ddot{\sigma}^m_{yy} \right) \right] \) evaluated at the upper and lower surfaces. Hence Equation (7.72) can be simplified

\[ i(\xi_m - \xi_r) 4P_{mn} = - \left[ \left( \ddot{\nu}^n_{xx} \sigma^m_{xy} + \ddot{\nu}^n_{yy} \sigma^m_{yy} \right) + \left( \nu^n_x \ddot{\sigma}^m_{xx} + \nu^n_y \ddot{\sigma}^m_{yy} \right) \right] \bigg|_{y=d}^{y=-d} \]  

(7.73)

where
\[ P_{mn} = \frac{1}{4} \int_{-d}^{d} \left[ (v^n_x \sigma^n_{xx} + v^n_y \sigma^n_{xy}) + (v^n_x \tilde{\sigma}^n_{xx} + v^n_y \tilde{\sigma}^n_{xy}) \right] dy \]  \hspace{1cm} (7.74)

If we assume free upper and lower surface, then the RHS of Equation (7.73) will vanish, because the stresses at \( y = d \) and \( y = -d \) are zero. Hence, under free surface conditions, we have

\[ i \left( \xi_m - \xi_n \right) P_{mn} = 0 \]  \hspace{1cm} (7.75)

So the orthogonality relation for the Lamb wave modes is

\[ P_{mn} = 0, \text{ when } m \neq n \]  \hspace{1cm} (7.76)

Recall the displacements and stress distribution of Lamb wave. The displacements are

\[ u_x = \left[ i \xi (A \cos \alpha y + B \sin \alpha y) + \beta (-G \sin \beta y + H \cos \beta y) \right] e^{i(\xi x - \omega t)} \]
\[ u_y = \left[ \alpha (-A \sin \alpha y + B \cos \alpha y) - i \xi (G \cos \beta y + H \sin \beta y) \right] e^{i(\xi x - \omega t)} \]  \hspace{1cm} (7.77)

where \( A, B, G, \) and \( H \) are constants determined by the Raleigh-Lamb frequency equations.

Hence the particle velocities are

\[ v_x = -i \omega \left[ i \xi (A \cos \alpha y + B \sin \alpha y) + \beta (-G \sin \beta y + H \cos \beta y) \right] e^{i(\xi x - \omega t)} \]
\[ v_y = -i \omega \left[ \alpha (-A \sin \alpha y + B \cos \alpha y) - i \xi (G \cos \beta y + H \sin \beta y) \right] e^{i(\xi x - \omega t)} \]

The stresses are

\[ \sigma_{xx} = \left[ -\left( \lambda + 2\mu \right) \xi^2 - \lambda \alpha^2 \right] (A \cos \alpha y + B \sin \alpha y) + 2i \mu \xi \beta (-G \sin \beta y + H \cos \beta y) \] \hspace{1cm} e^{i(\xi x - \omega t)}
\[ \sigma_{yy} = \left[ -\left( \lambda + 2\mu \right) \alpha^2 + \lambda \xi^2 \right] (A \cos \alpha y + B \sin \alpha y) - 2i \mu \xi \beta (-G \sin \beta y + H \cos \beta y) \] \hspace{1cm} e^{i(\xi x - \omega t)}
\[ \sigma_{xy} = \left[ 2i \mu \xi \alpha (-A \sin \alpha y + B \cos \alpha y) + \mu \left( \xi^2 - \beta^2 \right) (G \cos \beta y + H \sin \beta y) \right] e^{i(\xi x - \omega t)} \]
It is unwieldy to substitute the velocity and stress terms in Equation (7.74) and prove the orthogonality analytically. Instead, a program was used to calculate the $P_{nn}$ values numerically.

7.3.2 Lamb Modes Normalization

Lamb modes normalization consists of adjusting the amplitude of each mode such that the $P_{nm}$ values on the diagonal are unity. To achieve this, we divide the constants of Equations (4.9) by the square root of $P_{nn}$ such that

$$A_{new} = \frac{A_{old}}{\sqrt{P_{nn}^{old}}}$$

Upon application of this process, the normalized $P_{nn}$ values can be obtained.
CHAPTER 8 GENERAL GUIDED WAVES EQUATIONS

Consider a plate with free surfaces, with a straight-crested time harmonic wave propagating in \( +x \) direction (Figure 8.1).

\[ y = +d \]
\[ y = -d \]
\[ x \]
\[ z \]

Free surface
\[ \sigma_{yy} = 0 \]
\[ \sigma_{yz} = 0 \]
\[ \sigma_{zz} = \sigma_{xx} \]

Figure 8.1 (a) A plate with thickness 2\( b \), extends infinitely in \( x \) and \( z \) direction; (b) free body diagram of a small block on the surface of the plate.

\[ \Phi = f(y) e^{i(\xi x - \omega t)} \]
\[ H_x = h_x(y) e^{i(\xi x - \omega t)} \]
\[ H_y = h_y(y) e^{i(\xi x - \omega t)} \]
\[ H_z = h_z(y) e^{i(\xi x - \omega t)} \] \hspace{1cm} (8.1)

The governing equations are

\[ \mathbf{u} = \nabla \Phi + \nabla \times \mathbf{H} , \quad \nabla \cdot \mathbf{H} = 0 \] \hspace{1cm} (8.2)

\[ \nabla^2 \Phi = \frac{1}{c_p^2} \frac{\partial^2 \Phi}{\partial t^2} \]
\[ \nabla^2 \mathbf{H} = \frac{1}{c_s^2} \frac{\partial^2 \mathbf{H}}{\partial t^2} \] \hspace{1cm} (8.3)
where \( \nabla \Phi = \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y} + \frac{\partial \Phi}{\partial z} \), \( \nabla \cdot \mathbf{H} = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} \), and

\[
\nabla \times \mathbf{H} = \begin{vmatrix}
\bar{x} & \bar{y} & \bar{z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
H_x & H_y & H_z
\end{vmatrix} = \left( \frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial z} \right) \bar{x} + \left( \frac{\partial H_y}{\partial z} - \frac{\partial H_z}{\partial x} \right) \bar{y} + \left( \frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial y} \right) \bar{z}
\]

A \( z \)-invariant (plane strain) condition is assumed, which gives \( \frac{\partial}{\partial z} = 0 \). Hence, the displacements of Equation (4.1) simplify to

\[
\begin{align*}
\alpha_x &= \frac{\partial \Phi}{\partial x} + \frac{\partial H_z}{\partial y} \\
\alpha_y &= \frac{\partial \Phi}{\partial y} - \frac{\partial H_z}{\partial x} \\
\alpha_z &= -\frac{\partial H_x}{\partial y} + \frac{\partial H_y}{\partial x}
\end{align*}
\]

(8.4)

Substituting Equation (4.5) into (4.4), yields the following four equations

\[
\begin{align*}
\ddot{f} - \zeta^2 f &= \frac{1}{c_p^2} (-\omega^2 f) \\
\ddot{h}_x - \zeta^2 h_x &= \frac{1}{c_s^2} (-\omega^2 h_x) \\
\ddot{h}_y - \zeta^2 h_y &= \frac{1}{c_s^2} (-\omega^2 h_y) \\
\ddot{h}_z - \zeta^2 h_z &= \frac{1}{c_s^2} (-\omega^2 h_z)
\end{align*}
\]

(8.5)

Hence, the general solutions can be given in the form
\[
\Phi = (A \cos \alpha y + B \sin \alpha y)e^{i(\xi x - \omega t)} \\
H_x = (C \cos \beta y + D \sin \beta y)e^{i(\xi x - \omega t)} \\
H_y = (E \cos \beta y + F \sin \beta y)e^{i(\xi x - \omega t)} \\
H_z = (G \cos \beta y + H \sin \beta y)e^{i(\xi x - \omega t)}
\]

(8.6)

where \( \alpha^2 = \frac{\omega^2}{c_p^2} - \xi^2 \), and \( \beta^2 = \frac{\omega^2}{c_s^2} - \xi^2 \)

Substitution of Equations (4.8) into Equation (4.3) yields the displacements

\[
u_x = \frac{\partial \Phi}{\partial x} + \frac{\partial H_x}{\partial y} = [i\xi(A \cos \alpha y + B \sin \alpha y) + \beta(-G \sin \beta y + H \cos \beta y)]e^{i(\xi x - \omega t)}
\]

(8.7)

\[
u_y = \frac{\partial \Phi}{\partial y} - \frac{\partial H_x}{\partial x} = [\alpha(-A \sin \alpha y + B \cos \alpha y) - i\xi(G \cos \beta y + H \sin \beta y)]e^{i(\xi x - \omega t)}
\]

Recall the stresses are given in the form (Graff, 1975):

\[
\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}
\]

(8.8)

\[
\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})
\]

When the plane strain condition is applied, the stresses are simplified to

\[
\sigma_{xx} = (\lambda + 2\mu) \frac{\partial u_x}{\partial x} + \lambda \frac{\partial u_y}{\partial y}
\]

\[
\sigma_{yy} = (\lambda + 2\mu) \frac{\partial u_y}{\partial y} + \lambda \frac{\partial u_x}{\partial x}
\]

\[
\sigma_{xy} = \mu \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right)
\]

\[
\sigma_{yx} = \mu \frac{\partial u_x}{\partial y}
\]

(8.9)

Substituting Equation (4.9) into Equation (8.9) gives
\[ \sigma_{xx} = -\left[ (\lambda + 2\mu)\xi^2 - \lambda\alpha^2 \right] \left( A\cos\alpha y + B\sin\alpha y \right) + 2i\mu\xi\beta(-G\sin\beta y + H\cos\beta y) \]
\[ \sigma_{xy} = -\left[ (\lambda + 2\mu)\alpha^2 + \lambda\xi^2 \right] \left( A\cos\alpha y + B\sin\alpha y \right) - 2i\mu\xi\beta(-G\sin\beta y + H\cos\beta y) \]
\[ \sigma_{yx} = 2i\mu\xi\alpha(-A\sin\alpha y + B\cos\alpha y) + \mu\left( \xi^2 - \beta^2 \right)(G\cos\beta y + H\sin\beta y) \]
\[ \sigma_{yy} = \mu\beta^2 \left( C\cos\beta y + D\sin\beta y \right) + i\mu\xi\beta(-E\sin\beta y + F\cos\beta y) \]

The stress-free surfaces conditions at the upper and lower plate surfaces are

\[ \sigma_{yy} = \sigma_{xy} = \sigma_{yx} = 0, \quad y = \pm d \]  \hspace{1cm} (8.11)

Figure 8.2  
Free body diagram of stress-free boundary condition

This yields six equations. To solve for the eight unknowns (from A to H), the divergence condition \( \nabla \cdot \mathbf{H} = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = 0 \) is applied at the upper and lower surfaces, which yields the other two equations. The eight equations are

\[ \left[ (\lambda + 2\mu)\alpha^2 + \lambda\xi^2 \right] \left( A\cos\alpha d + B\sin\alpha d \right) + 2i\mu\xi\beta(-G\sin\beta d + H\cos\beta d) = 0 \]
\[ \left[ (\lambda + 2\mu)\alpha^2 + \lambda\xi^2 \right] \left( A\cos\alpha d - B\sin\alpha d \right) + 2i\mu\xi\beta(G\sin\beta d + H\cos\beta d) = 0 \]
\[ 2i\xi\alpha(-A\sin\alpha d + B\cos\alpha d) + \left( \xi^2 - \beta^2 \right)(G\cos\beta d + H\sin\beta d) = 0 \]
\[ 2i\xi\alpha(A\sin\alpha d + B\cos\alpha d) + \left( \xi^2 - \beta^2 \right)(G\cos\beta d - H\sin\beta d) = 0 \]  \hspace{1cm} (8.12)
\[ \beta^2 \left( C\cos\beta d + D\sin\beta d \right) + i\xi\beta(-E\sin\beta d + F\cos\beta d) = 0 \]
\[ \beta^2 \left( C\cos\beta d - D\sin\beta d \right) + i\xi\beta(E\sin\beta d + F\cos\beta d) = 0 \]
\[ i\xi(C\cos\beta d + D\sin\beta d) + \beta(-E\sin\beta d + F\cos\beta d) = 0 \]
\[ i\xi(C\cos\beta d - D\sin\beta d) + \beta(E\sin\beta d + F\cos\beta d) = 0 \]
by letting

\[ c_1 = (\lambda + 2\mu)\alpha^2 + \lambda \xi^2 \]
\[ c_2 = 2i\mu\xi\beta \]
\[ c_3 = 2i\xi\alpha \]
\[ c_4 = \xi^2 - \beta^2 \]
\[ c_5 = i\xi\beta \]

(8.13)

Equation (8.12) can be written in matrix form

\[
\begin{bmatrix}
   c_1\cos\alpha d & c_1\sin\alpha d & 0 & 0 & 0 & 0 & -c_2\sin\beta d & c_2\cos\beta d \\
   c_1\cos\alpha d & -c_1\sin\alpha d & 0 & 0 & 0 & 0 & -c_1\sin\beta d & c_1\cos\beta d \\
   -c_2\sin\alpha d & c_2\cos\alpha d & 0 & 0 & 0 & 0 & c_2\cos\beta d & c_2\sin\beta d \\
   c_2\sin\alpha d & c_2\cos\alpha d & 0 & 0 & 0 & 0 & -c_1\cos\beta d & -c_1\sin\beta d \\
   0 & 0 & \beta^2\cos\beta d & \beta^2\sin\beta d & -c_1\sin\beta d & c_1\cos\beta d & 0 & 0 \\
   0 & 0 & \beta\cos\beta d & -\beta^2\sin\beta d & c_1\sin\beta d & c_1\cos\beta d & 0 & 0 \\
   0 & 0 & i\xi\cos\beta d & i\xi\sin\beta d & -\beta\sin\beta d & \beta\cos\beta d & 0 & 0 \\
   0 & 0 & i\xi\cos\beta d & -i\xi\sin\beta d & \beta\sin\beta d & \beta\cos\beta d & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
   A \\
   B \\
   C \\
   D \\
   E \\
   F \\
   G \\
   H \\
\end{bmatrix} = 0
\]  

(8.14)

To simplify the matrix, the following steps are taken on the matrix rows R1, R2, R3, ..., R8 and columns C1, C2, C3, ..., C8

1. R2+R1, R4+R3, R6+R5, R8+R7
2. R1-R2, R3-R4, R5-R6, R7-R8
3. R1, R3 interchange, R6, R7 interchange
4. C2, C8 interchange, C4, C6 interchange
5. C4, C7 interchanger, C3, C8 interchange, C7, C8 interchange

The final matrix is
\[
\begin{bmatrix}
-c3\sin\alpha & c4\sin\beta & 0 & 0 & 0 & 0 & 0 & 0 \\
 c1\cos\alpha & c2\cos\beta & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & c1\sin\alpha & -c2\sin\beta & 0 & 0 & 0 & 0 \\
 0 & 0 & c3\cos\alpha & c4\cos\beta & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & -c5\sin\beta & \beta^2\sin\beta & 0 & 0 \\
 0 & 0 & 0 & 0 & -\beta\sin\beta & i\xi\sin\beta & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & \beta^2\cos\beta & c5\cos\beta \\
 0 & 0 & 0 & 0 & 0 & 0 & i\xi\cos\beta & \beta\cos\beta \\
\end{bmatrix}
\begin{bmatrix}
A \\
B \\
C \\
D \\
E \\
F \\
\end{bmatrix} = 0
\] (8.15)

From Equation (8.15) four sets of equations are obtained:

\[
\begin{bmatrix}
\beta^2\cos\beta & c5\cos\beta \\
i\xi\cos\beta & \beta\cos\beta \\
\end{bmatrix}
\begin{bmatrix}
C \\
F \\
\end{bmatrix} = 0 \quad C, F \neq 0
\] (8.16)

\[
\begin{bmatrix}
-c5\sin\beta & \beta^2\sin\beta \\
-\beta\sin\beta & i\xi\sin\beta \\
\end{bmatrix}
\begin{bmatrix}
E \\
D \\
\end{bmatrix} = 0 \quad E, D \neq 0
\] (8.17)

\[
\begin{bmatrix}
c1\sin\alpha & -c2\sin\beta \\
c3\cos\alpha & c4\cos\beta \\
\end{bmatrix}
\begin{bmatrix}
B \\
G \\
\end{bmatrix} = 0 \quad B, G \neq 0
\] (8.18)

\[
\begin{bmatrix}
-c3\sin\alpha & c4\sin\beta \\
c1\cos\alpha & c2\cos\beta \\
\end{bmatrix}
\begin{bmatrix}
A \\
H \\
\end{bmatrix} = 0 \quad A, H \neq 0
\] (8.19)

In each case, the constants not mentioned are assumed to be zero.

As we will see, the first two are the solutions for shear horizontal (SH) waves; while last two are for Lamb waves.

8.1 SH WAVE

8.1.1 Antisymmetric SH waves

Recall Equation (8.16)

\[
\begin{bmatrix}
\beta^2\cos\beta & c5\cos\beta \\
i\xi\cos\beta & \beta\cos\beta \\
\end{bmatrix}
\begin{bmatrix}
C \\
F \\
\end{bmatrix} = 0
\]
\[ u_z = \left[ (\beta C + i \xi F) \sin \beta y \right] e^{i(\xi - at)} \]
\[ u_x = u_y = 0 \] (8.20)

the determinant \[ \begin{vmatrix} \beta^2 \cos \beta d & c5 \cos \beta d \\ i \xi \cos \beta d & \beta \cos \beta d \end{vmatrix} = 0 \] yields

\[ \beta \left( \beta^2 + \xi^2 \right) \cos^2 \beta d = 0 \] (8.21)

The non-trivial solution is

\[ \beta d = (2n + 1) \frac{\pi}{2}, \quad n = 0, 1, 2, ... \] (8.22)

The first row of the matrix yields \( \beta (\beta C + i \xi F) \cos \beta d = 0 \), so that the coefficient \((\beta C + i \xi F)\) can take arbitrary value. Hence the displacements can be updated:

\[ u_z = [A_1 \sin \beta y] e^{i(\xi - at)} \]
\[ u_x = u_y = 0 \] (8.23)

where \( A_1 \) is an arbitrary constant

8.1.2 Symmetric SH waves

Recall Equation (8.17)

\[ \begin{bmatrix} -c5 \sin \beta d & \beta^2 \sin \beta d \\ -\beta \sin \beta d & i \xi \sin \beta d \end{bmatrix} \begin{bmatrix} E \\ D \end{bmatrix} = 0 \]

\[ u_z = \left[ (-\beta D + i \xi E) \cos \beta y \right] e^{i(\xi - at)} \]
\[ u_x = u_y = 0 \] (8.24)

the determinant \[ \begin{vmatrix} -c5 \sin \beta d & \beta^2 \sin \beta d \\ -\beta \sin \beta d & i \xi \sin \beta d \end{vmatrix} = 0 \] yields
\[
\beta \left( \beta^2 + \xi^2 \right) \sin^2 \beta d = 0 \tag{8.25}
\]

The non-trivial solution is
\[
\beta d = (2n) \frac{\pi}{2}, \quad n = 0, 1, 2, \ldots \tag{8.26}
\]

The first row of the matrix yields \(-\beta \left( i \xi E - \beta D \right) \sin \beta d = 0\), so that the coefficient \((-\beta D + i \xi E\)) can take arbitrary value. Hence the displacements can be updated:
\[
\begin{align*}
\vec{u}_x &= \begin{bmatrix} A_z \cos \beta y \end{bmatrix} e^{i(\xi x - \omega t)} \\
\vec{u}_y &= 0
\end{align*} \tag{8.27}
\]

where \(A_z\) is an arbitrary constant.

8.2 Lamb Waves

Antisymmetric Lamb waves

Recall Equation (8.18)
\[
\begin{bmatrix}
  c_1 \sin \alpha d & -c_2 \sin \beta d \\
  c_3 \cos \alpha d & c_4 \cos \beta d
\end{bmatrix}
\begin{bmatrix}
  B \\
  G
\end{bmatrix} = 0
\]

\[
\begin{align*}
\vec{u}_x &= \begin{bmatrix} i \xi B \sin \alpha y - \beta G \sin \beta y \end{bmatrix} e^{i(\xi x - \omega t)} \\
\vec{u}_y &= \begin{bmatrix} \alpha B \cos \alpha y - i \xi G \cos \beta y \end{bmatrix} e^{i(\xi x - \omega t)} \\
\vec{u}_z &= 0
\end{align*} \tag{8.28}
\]

the determinant
\[
\begin{vmatrix}
  c_1 \sin \alpha d & -c_2 \sin \beta d \\
  c_3 \cos \alpha d & c_4 \cos \beta d
\end{vmatrix} = 0 \quad \text{yields}
\]

\[
- \left( \xi^2 - \beta^2 \right)^2 \sin \alpha d \cos \beta d - 4\alpha \beta \xi^2 \sin \beta d \cos \alpha d = 0 \tag{8.29}
\]
This is the Rayleigh-Lamb frequency equation for the antisymmetric wave modes, and
normally written in the format of

$$\frac{-\left(\xi^2 - \beta^2\right)^2}{4\alpha\beta\xi^2} = \frac{\tan \beta d}{\tan \alpha d}$$  \hspace{1cm} (8.30)

The ratio of constants $B$ and $G$ can also be determined as

$$\frac{B}{G} = \frac{-2i\xi\beta \sin \beta d}{\left(\xi^2 - \beta^2\right) \sin \alpha d} = -\frac{\left(\xi^2 - \beta^2\right) \cos \beta d}{2i\xi \alpha \cos \alpha d}$$  \hspace{1cm} (8.31)

Symmetric Lamb waves

Recall Equation (8.19)

$$\begin{bmatrix} -c_3 \sin \alpha d & c_4 \sin \beta d \\ c_1 \cos \alpha d & c_2 \cos \beta d \end{bmatrix} \begin{bmatrix} A \\ H \end{bmatrix} = 0$$

$$u_x = [i\xi A \cos \alpha y + \beta H \cos \beta y] e^{i(\xi x - \omega t)}$$

$$u_y = [-\alpha A \sin \alpha y - i\xi H \sin \beta y] e^{i(\xi x - \omega t)}$$

$$u_z = 0$$

the determinant $\begin{vmatrix} -c_3 \sin \alpha d & c_4 \sin \beta d \\ c_1 \cos \alpha d & c_2 \cos \beta d \end{vmatrix} = 0$ yields

$$4\xi^2 \alpha \beta \sin \alpha d \cos \beta d + \left(\xi^2 - \beta^2\right)^2 \sin \beta d \cos \alpha d = 0$$  \hspace{1cm} (8.33)

This is the Rayleigh-Lamb frequency equation for the symmetric wave modes, and
normally written in the format of

$$\frac{-4\alpha\beta\xi^2}{\left(\xi^2 - \beta^2\right)^2} = \frac{\tan \beta d}{\tan \alpha d}$$  \hspace{1cm} (8.34)
The ratio of constants $B$ and $G$ can also be determined as

$$
\frac{A}{H} = \frac{2i\xi\beta \cos \beta d}{(\xi^2 - \beta^2) \cos \alpha d} = \frac{(\xi \beta - \beta^2) \sin \beta d}{2i\xi\alpha \sin \alpha d}
$$

(8.35)
PART II  PWAS LAMB WAVE TRANSDUCERS
CHAPTER 9 STATE OF THE ART IN PWAS LAMB WAVE TRANSDUCERS

PWAS are made from piezoelectric materials. Piezoelectricity (discovered in 1880 by Jacques and Pierre Curie) describes the phenomenon of generating an electric field when the material is subjected to a mechanical stress (direct effect), or, conversely, generating a mechanical strain in response to an applied electric field. The direct piezoelectric effect, predicts how much electric field is generated by a given mechanical stress. This sensing effect is utilized in the development of piezoelectric sensors. The converse piezoelectric effect predicts how much mechanical strain is generated by a given electric field. This actuation effect is utilized in the development of piezoelectric induced-strain actuators.

Piezoelectric properties occur naturally in some crystalline materials, e.g., quartz crystals (SiO2), and Rochelle salt. The latter is a natural ferroelectric material, possessing an orientable domain structure that aligns under an external electric field and thus enhances its piezoelectric response. Piezoelectric response can also be induced by electrical poling certain polycrystalline materials, such as piezoceramics.

Commonly used conventional Lamb wave transducers introduce an angled incident wave, and use the interference of the incident wave and the reflected wave to generate Lamb waves (Rose, 1999, pp106). Some other methods used comb transducer (Viktorov, 1967) or interdigital transducers (Ditri, Rose, and Pilarski, 1993) which introduce surface distributed stresses to generate Lamb waves.

PWAS generate and detect Lamb waves by utilizing the direct and converse piezoelectric effects. Because of its small size and high efficiency, more and more researchers are
getting interested in the study of PWAS Lamb wave transducers. Pierce et. al. (1997), Culshaw, Pierce, Staszekski (1998), and Gachagan (1999) introduced a Structurally Integrated System for the comprehensive evaluation of Composites (SISCO). It was an acoustic/ultrasonic based structural monitoring system for composite structures. They studied a low profile acoustic source (LPAS) in this system. The LPAS used a 1-3 connectivity piezo-composite layer (Figure 9.1a) as the active phase and two flexible printed circuit boards (PCB) as the upper and lower electrodes. The PCB was designed with a interdigital (ID) electrode pattern. (Figure 9.1b)

![Diagram of LPAS](image)

**Figure 9.1** Construction of 1-3 connectivity piezo-composite (Pierce, S. G., et. al., 1997) and the interdigital pattern on the electrode of LPAS transducer (Gachagan, A., et. al., 1999)

The spacing of the ID pattern was designed to be one half of the desired Lamb wavelength and the fingers of the ID pattern were driven differentially. The overall thickness of the LPAS was approximately 0.7 mm

Lemistre, Osmont, and Balageas (2000) presented an active health system based on wavelet transform analysis. Multiple 5 mm diameter 0.1 mm thick PZT (lead zirconate titanate) discs were used as both Lamb wave transmitter and receivers. They were mounted on the surface of a carbon-epoxy composite plate.
Osmont et al. (2000) presented damage and damaging impact monitoring system using PZT sensors. Multiple 5 mm diameter 0.1 mm thick PZT discs were used as sensors. These PZT discs were mounted (a) inside and (b) on the surface of carbon-epoxy plates.

![Image of PZT patches bonded to beam](image)

**Figure 9.2**  Two PZT patches (20x5 mm) were bonded close to the end of a narrow carbon-epoxy composite beam to detect the controlled delamination at the center of the beam. One of the PZT patch is used as actuator and the other as sensor. (Diaz Valdes, and Soutis, 2000)

Diaz Valdes and Soutis (2000) demonstrated an application of PZT patch in detecting delamination in a Carbon-epoxy composite beam. Two 20x5 mm PZT patches were made from commercial brass backed piezoceramic resonators. They were bonded on the surface of the beam close to one end. (Figure 9.2). One of the two PZT patches was used as transducer, and the other was used as sensor.

Lin and Yuan (2001b) studied the modeling of diagnostic transient wave in an integrated piezoelectric sensor/actuator plate. PZT ceramic disks were mounted on the surface of an aluminum plate acting as both actuators and sensors to generate and collect $A_0$ mode Lamb waves.

The experimental setup up and results of these introduced PWAS Lamb wave transducers will be reviewed in Chapter 12 of this dissertation.

In Part II of this dissertation, piezoelectric effects is discussed. PWAS actuation and sensing are modeled. The case of a PWAS bonded to a thin plate is discussed.

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CHAPTER 10     PIEZOELECTRIC WAFER ACTIVE SENSORS

10.1 ACTUATION EQUATIONS

For linear piezoelectric materials, the interaction between the electrical and mechanical variables can be described by linear relations (ANSI/IEEE Standard 176-1987). A constitutive relation is established between mechanical and electrical variables in the tensorial form:

\[ S_{ij} = s_{ijkl}^T T_{kl} + d_{ijk} E_k + \delta_{ij} \alpha_i^T \theta \]  
\[ D_j = d_{jkl} T_{kl} + \varepsilon_{jk}^T E_k + \tilde{D}_j \theta \]

(10.1)  (10.2)

the symbols used in these constitutive relations are:

- \( S_{ij} \) mechanical strain
- \( T_{ij} \) mechanical stress
- \( E_i \) electrical field
- \( D_i \) electrical displacement (charge per unit area)
- \( s_{ijkl}^T \) mechanical compliance of the material measured at zero electric field \((E = 0)\)
- \( \varepsilon_{jk}^T \) dielectric constant measured at zero mechanical stress \((T = 0)\)
$d_{ij}$ piezoelectric strain constant (also known as the piezoelectric charge constant),
couples the electrical and mechanical variables and expresses how much strain
is obtained per unit applied electric field

$\theta$ Temperature

$\alpha_i$ coefficient of thermal expansion under constant electric field

$\tilde{D}_j$ electric displacement temperature coefficient

The stress and strain variables are second order tensors, while the electric field and the
electric displacement are first order tensors. Since thermal effects only influence the
diagonal terms, the respective coefficients, $\alpha_i$ and $\tilde{D}_j$, have single subscripts. The term
$\delta_{ij}$ is the Kroneker delta ($\delta_{ij} = 1$ if $i = j$; zero otherwise). The superscripts $T$, $D$, $E$
signify that the quantities are measured at zero stress ($T = 0$), zero electric displacement
($D = 0$), or zero electric field ($E = 0$), respectively. In practice, the zero electric
displacement condition corresponds to open circuit (zero current across the electrodes),
while the zero electric field corresponds to closed circuit (zero voltage across the
electrodes).

Equation (10.1) is the actuation equation. It is used to predict how much will be created at
a given stress, electric field, and temperature. The terms proportional with stress and
temperature are common with the formulations of classical thermoelasticity. The term
proportional with the electric field is specific to piezoelectricity and represents the
induced strain actuation (ISA), i.e.,
\[ S_{ij}^{*RA} = d_{ij}E_k \] (10.3)

Equation (10.2) is used to predict how much electric displacement, i.e., charge per unit area, is required to accommodate the simultaneous state of stress, electric field, and temperature.

In engineering practice, the tensorial Equation (10.1) is rearranged in matrix form using Voigt notations, in which the stress and strain tensors are arranged as 6-component vectors, where the first 3 components represent direct stress and strain, while the last three components represent shear stress and strain. Thus:

\[
\begin{bmatrix}
S_{11} \\
S_{22} \\
S_{33} \\
S_{23} \\
S_{31} \\
S_{12}
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{zz} \\
\varepsilon_{yz} \\
\varepsilon_{zx} \\
\varepsilon_{xy}
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
T_{11} \\
T_{22} \\
T_{33} \\
T_{23} \\
T_{31} \\
T_{12}
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\sigma_{yz} \\
\sigma_{zx} \\
\sigma_{xy}
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5 \\
\varepsilon_6
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6
\end{bmatrix}
\] (10.4)

Hence, the constitutive Equation (1) takes the matrix form:

\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5 \\
\varepsilon_6
\end{bmatrix} =
\begin{bmatrix}
s_{11} & s_{12} & s_{13} \\
s_{21} & s_{22} & s_{23} \\
s_{31} & s_{32} & s_{33}
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6
\end{bmatrix}
+ \begin{bmatrix}
d_{11} & d_{21} & d_{31} \\
d_{12} & d_{22} & d_{32} \\
d_{13} & d_{23} & d_{33} \\
d_{14} & d_{24} & d_{34} \\
d_{15} & d_{25} & d_{35} \\
d_{16} & d_{26} & d_{36}
\end{bmatrix}
\begin{bmatrix}
E_1 \\
E_2 \\
E_3
\end{bmatrix} + \begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{bmatrix} \Delta T
\] (10.5)

i.e.,

\[ \{\varepsilon\} = [s]\{\sigma\} + [d]\{E\} + \{\alpha\} \Delta T \] (10.6)

In these equations, \(\Delta T\) is the temperature difference.
In practical applications, many of the piezoelectric coefficients, $d_{ij}$, have negligible values, since the piezoelectric materials respond preferentially along certain directions. Customary, only $d_{13}$, $d_{23}$, $d_{33}$, $d_{15}$, $d_{25}$ are non-zero. In addition, symmetry of the compliance matrix is implied. Hence, the following equations apply:

$$
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5 \\
\varepsilon_6 \\
\end{bmatrix} =
\begin{bmatrix}
s_{11} & s_{12} & s_{13} \\
s_{21} & s_{22} & s_{23} \\
s_{31} & s_{32} & s_{33} \\
s_{44} & s_{55} & s_{66} \\
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6 \\
\end{bmatrix} +
\begin{bmatrix}
d_{31} \\
d_{32} \\
d_{33} \\
d_{15} \\
\end{bmatrix}
\begin{bmatrix}
E_1 \\
E_2 \\
E_3 \\
\end{bmatrix} +
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\end{bmatrix} \Delta T \ (10.7)
$$

For piezoelectric materials with transverse isotropy, these equations simplify further by taking $d_{23} = d_{13}$, $d_{25} = d_{15}$. Hence, for transverse isotropy, the constitutive actuation equations become:

$$
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5 \\
\varepsilon_6 \\
\end{bmatrix} =
\begin{bmatrix}
s_{11} & s_{12} & s_{13} \\
s_{12} & s_{11} & s_{13} \\
s_{13} & s_{13} & s_{33} \\
s_{44} & s_{55} & s_{66} \\
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6 \\
\end{bmatrix} +
\begin{bmatrix}
d_{31} \\
d_{31} \\
d_{33} \\
d_{15} \\
\end{bmatrix}
\begin{bmatrix}
E_1 \\
E_1 \\
E_1 \\
\end{bmatrix} +
\begin{bmatrix}
\alpha_1 \\
\alpha_1 \\
\alpha_3 \\
\end{bmatrix} \Delta T \ (10.8)
$$

Figure 10.1 illustrates the physical meaning of these equations in the case of simple polarization. Figure 10.1a shows that an electric field $E_3 = V/t$ applied parallel to the direction of polarization ($E_3 \parallel P$) induces thickness expansion ($\varepsilon_{33} = d_{33}E_3$) and transverse contraction ($\varepsilon_1 = d_{11}E_3$). Figure 10.1b shows that, if the field is perpendicular to the direction of polarization ($E_3 \perp P$), shear deformation is induced ($\varepsilon_5 = d_{15}E_3$).
Figure 10.1  Basic induc-strain responses of piezoelectric materials: (a) axial and transverse
strain; (b) shear strain (after Piezo Systems, Inc.)

The values of the piezoelectric coupling coefficients, $d_{31}$, $d_{32}$, $d_{33}$, $d_{45}$, differ from
material to material. In certain crystalline piezoelectric materials, the piezoelectric
coefficients, $d_{ij}$, ($i = 1, \ldots, 6; j = 1, 2, 3$) may be enhanced or diminished through preferred
crystal cut orientation. The various possible crystal cuts for piezoelectric quartz are
shown in Figure 10.2. The QCM (quartz crystal microbalance) sensor uses the AT cut
which enhances the shear strain response ($d_{51}$ piezoelectric coefficient).

Figure 10.2  Directional dependence of quartz crystal piezoelectricity: (a) crystallographic cuts of
quartz crystals; (b) practical cuts of quartz resonators (after Ikeda, 1996)
10.2 SENSING EQUATIONS

So far, the piezoelectric equations have expressed the strain and electric displacement in terms of applied stress, electric field, and temperature using the constitutive tensorial Equations (10.1) and (10.2), and their matrix correspondents. However, these equations can be replaced by an equivalent set of equations that highlight the sensing effect, i.e., predict how much electric field will be generated by a given state of stress, electric displacement, and temperature. Thus, Equations (10.1) and (10.2) can be expressed as:

\[ S_{ij} = s^{D}_{ijkl} T_{kl} + g_{ijk} D_k + \delta_{ij} \alpha_i^{D}\theta \]  
\[ E_j = g_{jkl} T_{kl} + \beta^{T}_{jk} D_k + \bar{E}_j \theta \]

Equation (10.10) predicts how much electric field, i.e., voltage per unit thickness, is generated by “squeezing” the piezoelectric material, i.e., represents the direct piezoelectric effect. This formulation is useful in piezoelectric sensor design. Equation (10.10) is called the sensor equation. The coefficient \( g_{ijk} \) is the piezoelectric voltage constant, and represent how much electric field is induced per unit stress. The coefficients \( \bar{E}_j \) are the pyroelectric voltage coefficients and represent how much electric field is induced per unit temperature change.

10.3 STRESS EQUATIONS

The piezoelectric constitutive equations can also be expressed in such a way as to reveal stress and electric displacement in terms of strain and electric field. This formulation is especially useful for incorporating the piezoelectric constitutive equations in stress and strength analyses. The stress formulation of the piezoelectric constitutive equations are:
\[ T_i = c^{E}_{ijkl} S_{kl} - e_{ijk} E_k - c^{E}_{ijkl} \alpha_k^E \theta \]  

\[ D_i = e_{ik} T_{kl} + \theta^T E_k + \tilde{D}_i \theta \]  

where \( c^{E}_{ijkl} \) is the stiffness tensor, and \( e_{ijk} \) is the piezoelectric stress constant. The term \( -c^{E}_{ijkl} \alpha_k^E \theta \) represents the stress induced in a piezoelectric material by temperature changes when the strain is forced to be zero. This corresponds, for example, the material being fully constraint against deformation. Such stresses, which are induced by temperature effects, are also known as residual thermal stresses. They are very important in calculation the strength of piezoelectric materials, especially when they are processed at elevated temperatures.

### 10.4 Actuator Equations in Terms of Polarization

In practical piezoelectric sensor and actuator design, the use of electric field, \( E_i \), and electric displacement, \( D_j \), is more convenient, since these variables relate directly to the voltage and current that can experimentally measured. However, theoretical explanations of the observed phenomena using solid-state physics are more direct when the polarization \( P_j \) is used instead of the electric displacement, \( D_j \). The polarization, electric displacement, and electric field are related by:

\[ D_i = \varepsilon_0 E_i + P_i \]  

where, \( \varepsilon_0 \) is the free space dielectric permittivity. On the other hand, the electric field and electric displacement are related by:

\[ D_i = \varepsilon_j E_j \]
Here, ε_ij is, by definition, the effective dielectric permittivity of the material. Thus, the polarization can be related to the electric field in the form:

$$P_i = (\varepsilon_{ij} - \varepsilon_0)E_j = \kappa_{ij}E_j$$  \hspace{1cm} (10.15)

In terms of polarization, P_i, Equations (10.2) can be expressed in the form:

$$S_{ij} = s^{E}_{ijkl}T_{kl} + d_{ijk}E_k + \delta_{ij}\alpha_0\theta$$  \hspace{1cm} (10.16)

$$P_j = d_{jkl}T_{kl} + \kappa_{jkl}^T E_k + \bar{P}_j \theta$$  \hspace{1cm} (10.17)

where $\bar{P}_j$ is the coefficient of pyroelectric polarization.

10.5 COMPRESSED MATRIX NOTATIONS FORMULATION

In order to write the elastic and piezoelectric tensors in the form of matrix arrays, a compressed matrix notation is introduced to replace the tensor notation (Voigt notations). This compressed matrix notation consists of replacing $ij$ or $kl$ by $p$ or $q$, where $i,j,k,l = 1,2,3$ and $p,q = 1,2,3,4,5,6$. Thus, the 3x3x3x3 fourth order stiffness and compliance tensors $c^{E}_{ijkl}$ and $s^{E}_{ijkl}$ are replaced by 6x6 stiffness and compliance matrices of elements $c^{E}_{pq}$ and $s^{E}_{pq}$. Similarly, the tensors $c^{D}_{ijkl}$ and $s^{D}_{ijkl}$ are replaced by 6x6 stiffness and compliance matrices of elements $c^{D}_{pq}$ and $s^{D}_{pq}$. The 3x3 piezoelectric tensors, $d_{ijkl}$, $e_{ijkl}$, $g_{ijkl}$, and $h_{ijkl}$, are replaced by 3x6 piezoelectric matrices of elements $d_{ip}$, $e_{ip}$, $g_{ip}$, $h_{ip}$. The 3x3 stress and strain tensors, $T_{ij}$ and $S_{ij}$, are replaced by 6-long column matrices of elements $T_p$ and $S_p$. The following rules apply:

$$T_p = T_{ij}, \quad p = 1,2,3,4,5,6 \quad \text{(Stress)}$$  \hspace{1cm} (10.18)
\[ S_p = S_y, \quad i = j, \quad p = 1,2,3 \] (Strain) \hspace{3cm} (10.19)

\[ S_p = 2S_y, \quad i \neq j, \quad p = 4,5,6 \]

The factor of two in the strain equation is related to a factor of 2 in the definition of shear strains in the tensor and matrix formulation.

\[ c_{pq}^E = c_{ijkl}^E, \quad c_{pq}^D = c_{ijkl}^D \quad p = 1,2,3,4,5,6 \] (Stiffness coefficients) \hspace{1cm} (10.20)

\[
\begin{align*}
s_{pq}^E &= s_{ijkl}^E, & i = j \text{ and } k = l, & p,q = 1,2,3 \\
s_{pq}^E &= 2s_{ijkl}^E, & i = j \text{ and } k \neq l, & p = 1,2,3, \quad q = 4,5,6 \\
s_{pq}^E &= 4s_{ijkl}^E, & i \neq j \text{ and } k \neq l, & p,q = 4,5,6
\end{align*}
\] (Compliance coefficients)\hspace{1cm}(10.21)

and similar for \( s_{pq}^D \). The factors of 2 and 4 are associated with the factor of 2 from the strain equations.

\[ e_{qp} = e_{ijkl}, \quad h_{qp} = h_{ijkl} \] (Piezoelectric stress constants) \hspace{3cm} (10.22)

\[
\begin{align*}
d_{pq} &= d_{ijkl}, & k = l, & q = 1,2,3 \\
d_{pq} &= 2d_{ijkl}, & k \neq l, & q = 4,5,6
\end{align*}
\] (Piezoelectric strain constants) \hspace{1cm} (10.23)

\[
\begin{align*}
g_{pq} &= g_{ijkl}, & k = l, & q = 1,2,3 \\
g_{pq} &= 2g_{ijkl}, & k \neq l, & q = 4,5,6
\end{align*}
\] (Piezoelectric voltage constants) \hspace{1cm} (10.24)

The compressed matrix notations have the advantage of brevity. They are commonly used in engineering applications. The values of the elastic and piezoelectric constants given by the active material manufacturers in there product specifications are given in compressed matrix notations.
10.6 RELATIONS BETWEEN THE CONSTANTS

The constants that appear in the equations described in the previous sections can be related to each other. For example, the stiffness tensor, \( c_{ijkl} \), is the inverse of the strain tensor, \( s_{ijkl} \). Similar relations can be establish for the other constants and coefficients. In writing these relations, we use the compressed matrix notation with \( i,j,k,l = 1,2,3 \) and \( p,q,r = 1,2,3,4,5,6 \). We also use the 3x3 unit matrix \( \delta_{ij} \) and the 6x6 unit matrix \( \delta_{pq} \). As before, Einstein convention of implied summation over the repeated indices applies.

\[
e_{pq}^{E} s_{qr}^{E} = \delta_{pq}, \quad c_{pq}^{E} s_{qr}^{D} = \delta_{pq} \quad \text{(Stiffness—compliance relations)} \quad (10.25)
\]

\[
e_{ik}^{S} \beta_{jk}^{S} = \delta_{ij}, \quad \beta_{ik}^{T} e_{jk}^{T} = \delta_{ij} \quad \text{(Permittivity—impermittivity relations)} \quad (10.26)
\]

\[
c_{pq}^{D} = c_{pq}^{E} + e_{ik} h_{kq}, \quad s_{pq}^{D} = s_{pq}^{E} - d_{ik} g_{kq}
\]

(Close circuit—open circuit effects on elastic constants) \quad (10.27)

\[
e_{ij}^{T} = e_{ij}^{S} + d_{ik} e_{jk}^{S}, \quad \beta_{ij}^{T} = \beta_{ij}^{S} - g_{iq} h_{jq}
\]

(Stress-strain effects on dielectric constants) \quad (10.28)

\[
\begin{align*}
e_{ip} &= d_{iq} c_{qp}^{E}, \\
d_{ij} &= e_{ik}^{T} g_{jp} \\
g_{iq} &= \beta_{ik}^{T} d_{jq}, \\
h_{ip} &= g_{iq} e_{qp}^{D}
\end{align*}
\]

(Relations between piezoelectric constants) \quad (10.29)

10.7 ELECTROMECHANICAL COUPLING COEFFICIENT

Electromechanical coupling coefficient is defined as the square root of the ratio between the mechanical energy stored and the electrical energy applied to a piezoelectric material:

\[
k = \sqrt{\frac{\text{Mechanical energy stored}}{\text{Electrical energy applied}}} \quad (10.30)
\]
For direct actuation, we have \( k_{33} = \frac{d_{33}}{\sqrt{s_{33}e_{33}}} \), for transverse actuation, \( k_{31} = \frac{d_{31}}{\sqrt{s_{11}e_{33}}} \), and for shear actuation \( k_{15} = \frac{d_{15}}{\sqrt{s_{55}e_{11}}} \). For uniform inplane deformation, we obtain the planar coupling coefficient, \( \kappa_p = \kappa_{13} \sqrt{\frac{2}{1-\nu^2}} \), where \( \nu \) is the Poisson ratio.

10.8 Higher Order Models of the Electroactive Response

Higher order models of the electroactive ceramics contain both linear and quadratic terms. The linear terms are associated with the conventional piezoelectric response. The quadratic terms are associated with the electrostrictive response, whereas the application of electric field in one direction induces constriction (squeezing) of the material. The electrostrictive effect is not limited piezoelectric materials, and is present in all materials, though with different amplitudes. The electrostrictive response is quadratic in electric field. Hence, the direction of the electrostriction does not switch as the polarity of the electric field is switched. The constitutive equations that incorporate both piezoelectric and electrostrictive response have the form:

\[
S_{ij} = s_{ijkl}T_{kl} + d_{klj}E_k + M_{klji}E_kE_i, \tag{10.31}
\]

Note that the first two terms in the first equation are the same as for piezoelectric materials. The third term is due to electrostriction. The coefficients \( M_{klji} \) are the electrostrictive coefficients.
Table 10.1 Properties of piezoelectric ceramic APC-850 (www.americanpiezo.com)

<table>
<thead>
<tr>
<th>Property</th>
<th>APC 840</th>
<th>APC 841</th>
<th>APC 850</th>
<th>APC 850</th>
<th>APC 855</th>
<th>APC 856</th>
<th>APC 880</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$ (kg/m$^3$)</td>
<td>7600</td>
<td>7600</td>
<td>7700</td>
<td>7500</td>
<td>7500</td>
<td>7600</td>
<td></td>
</tr>
<tr>
<td>$d_{33}$ (10$^{-12}$m/V)</td>
<td>290</td>
<td>275</td>
<td>400</td>
<td>580</td>
<td>620</td>
<td>215</td>
<td></td>
</tr>
<tr>
<td>$d_{31}$ (10$^{-12}$m/V)</td>
<td>-125</td>
<td>109</td>
<td>-175</td>
<td>270</td>
<td>260</td>
<td>-95</td>
<td></td>
</tr>
<tr>
<td>$d_{15}$ (10$^{-12}$m/V)</td>
<td>480</td>
<td>450</td>
<td>590</td>
<td>720</td>
<td>710</td>
<td>330</td>
<td></td>
</tr>
<tr>
<td>$g_{33}$ (10$^{-3}$V/m/N)</td>
<td>26.5</td>
<td>25.5</td>
<td>26</td>
<td>19.5</td>
<td>18.5</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>$g_{31}$ (10$^{-3}$V/m/N)</td>
<td>-11</td>
<td>10.5</td>
<td>-12.4</td>
<td>8.8</td>
<td>8.1</td>
<td>-10</td>
<td></td>
</tr>
<tr>
<td>$g_{16}$ (10$^{-3}$V/m/N)</td>
<td>38</td>
<td>35</td>
<td>36</td>
<td>27</td>
<td>25</td>
<td>28</td>
<td></td>
</tr>
<tr>
<td>$s_{11}$ (10$^{-12}$ m$^2$/N)</td>
<td>11.8</td>
<td>11.7</td>
<td>15.3</td>
<td>14.8</td>
<td>15.0</td>
<td>10.8</td>
<td></td>
</tr>
<tr>
<td>$s_{33}$ (10$^{-12}$ m$^2$/N)</td>
<td>17.4</td>
<td>17.3</td>
<td>17.3</td>
<td>16.7</td>
<td>17.0</td>
<td>15.0</td>
<td></td>
</tr>
<tr>
<td>$\varepsilon_{33}^T / \varepsilon_0$</td>
<td>1250</td>
<td>1350</td>
<td>1750</td>
<td>3250</td>
<td>4100</td>
<td>1000</td>
<td></td>
</tr>
<tr>
<td>$k_p$</td>
<td>0.59</td>
<td>0.60</td>
<td>0.63</td>
<td>0.65</td>
<td>0.65</td>
<td>0.50</td>
<td></td>
</tr>
<tr>
<td>$k_{33}$</td>
<td>0.72</td>
<td>0.68</td>
<td>0.72</td>
<td>0.74</td>
<td>0.73</td>
<td>0.62</td>
<td></td>
</tr>
<tr>
<td>$k_{31}$</td>
<td>0.35</td>
<td>0.33</td>
<td>0.36</td>
<td>0.38</td>
<td>0.36</td>
<td>0.30</td>
<td></td>
</tr>
<tr>
<td>$k_{16}$</td>
<td>0.70</td>
<td>0.67</td>
<td>0.68</td>
<td>0.66</td>
<td>0.65</td>
<td>0.55</td>
<td></td>
</tr>
<tr>
<td>Poisson ratio, $\sigma$</td>
<td>0.30</td>
<td>0.40</td>
<td>0.35</td>
<td>0.32</td>
<td>0.39</td>
<td>0.28</td>
<td></td>
</tr>
<tr>
<td>Young’s Modulus $\gamma_{11}^E$ (GPa)</td>
<td>80</td>
<td>76</td>
<td>63</td>
<td>61</td>
<td>58</td>
<td>90</td>
<td></td>
</tr>
<tr>
<td>Young’s Modulus $\gamma_{33}^E$ (GPa)</td>
<td>68</td>
<td>63</td>
<td>54</td>
<td>48</td>
<td>45</td>
<td>72</td>
<td></td>
</tr>
<tr>
<td>Curie Temp. ($^\circ$C)</td>
<td>325</td>
<td>320</td>
<td>360</td>
<td>195</td>
<td>150</td>
<td>310</td>
<td></td>
</tr>
<tr>
<td>Dissipation factor, tan$\delta$ (%)</td>
<td>0.4</td>
<td>0.35</td>
<td>1.4</td>
<td>2</td>
<td>2.7</td>
<td>0.35</td>
<td></td>
</tr>
<tr>
<td>Mechanical $Q_m$</td>
<td>500</td>
<td>1400</td>
<td>80</td>
<td>75</td>
<td>72</td>
<td>1000</td>
<td></td>
</tr>
</tbody>
</table>

Note: Poisson ratio calculated from the formula: $k_p = \frac{1}{1 - \sigma} k_{31}$. 

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10.9 STRAIN DETECTION WITH PWAS

Figure 10.3 A PWAS wafer under strain in \( x_1 \) direction. (a) the direction vectors on the PWAS wafer; (b) the strain is in the \( x_i \) direction; (c) the electrical configuration having an external capacitor \( C \) connected in parallel to the internal capacitance.

In PWAS applications, the mechanical stress is applied in the \( x_i \) direction, and the electric field is in the \( x_3 \) direction. Let's consider an \( a = 7 \text{mm}, t = 0.2 \text{mm} \) PWAS made of APC 850 PZT wafer (properties in Table 10.2).

The constitutive equations are

\[
S_i = s_{11}^E T_1 + d_{31} E_3 \\
D_3 = d_{33} T_1 + e_{33}^T E_3
\]

(10.32)

Notice that

\[
\begin{align*}
q & = D_3 A \\
V & = q / C \\
\Rightarrow D_3 & = V \frac{C}{A}
\end{align*}
\]

(10.33)

where \( q \) is the charge on the surface of the PWAS and external capacitor, \( A \) is the surface area of the PWAS, and \( V \) is the voltage on the PWAS and the capacitor.

Also the electric field is given by
\[ E_3 = -\frac{V}{t} \]  \hspace{1cm} (10.34)

From Equations (10.32),

\[ T_1 = S_{i_1} \frac{d_{31}}{s_{i_1}} E_3 \]  \hspace{1cm} (10.35)

substitute the stress in Equation (10.32) with Equation (10.35)

\[ D_3 = d_{31} \left( \frac{S_{i_1}}{s_{i_1}} \frac{d_{31}}{s_{i_1}} E_3 \right) + \varepsilon^{\tau}_{33} E_3 \]  \hspace{1cm} (10.36)

substitute Equation (10.33) and (10.34) into (10.36)

\[ V \frac{C}{A} = d_{31} \left( \frac{S_{i_1}}{s_{i_1}} + \frac{d_{31}}{s_{i_1}} \frac{V}{t} \right) - \varepsilon^{\tau}_{33} \frac{V}{t} \]

collect \( V \) terms

\[ V \left( \frac{C}{A} + \frac{\varepsilon^{\tau}_{33}}{t} - \frac{1}{s_{i_1}} \frac{d_{31}}{t} \frac{1}{s_{i_1}} \right) = d \frac{S_{i_1}}{s_{i_1}} \]

hence, \( V \) is derived as

\[ V = \frac{d_{31} \frac{S_{i_1}}{s_{i_1}}}{C + \varepsilon^{\tau}_{33} \frac{d_{31}}{s_{i_1}} \frac{1}{t}} \quad \frac{d_{31} \frac{A}{s_{i_1}}}{C + \frac{A \varepsilon^{\tau}_{33}}{t} \left( 1 - \frac{d_{31}^2}{\varepsilon^{\tau}_{33} s_{i_1}} \right)} = \frac{d_{31} \frac{A}{s_{i_1}}}{C + c_0 \left( 1 - \kappa_{31}^2 \right)} \frac{S_{i_1}}{s_{i_1}} \]

where \( c_0 = \frac{A \varepsilon^{\tau}_{33}}{t} \) is the stress-free capacitance of the PWAS.

In our example, \( a = 7 \text{mm} \) and \( t = 0.2 \text{mm} \), hence

\[ c_0 = \frac{A \varepsilon^{\tau}_{33}}{t} = \frac{\left( 7 \cdot 10^{-3} \right)^2 \cdot 1750 \cdot \varepsilon_0}{0.2 \cdot 10^{-3}} = 3.796 \cdot 10^{-9} \text{ Farad} \]

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and the voltage on the electrode is

\[
V = \frac{175 \cdot 10^{-12} \left(7 \cdot 10^{-3}\right)^2}{C + 3.796 \cdot 10^{-9} \left(1 - 0.36^2\right)} \frac{5.605 \cdot 10^{-4}}{C + 3.304 \cdot 10^{-9}} S_i \text{ (volts)}
\]  

(10.37)

We can now examine the extreme situations. One would be: the external capacitor

\( C \gg C_0 \), and the other would be the \( C \leq C_0 \).

Case I: \( C \gg C_0 \) then Equation (10.37) can be simplified

\[
V \equiv \frac{5.605 \cdot 10^{-4}}{C} S_i \text{ (volts)}
\]

Assume \( C = 1 \cdot 10^{-6} \) Farad, then for every micro-strain,

\[
V = \frac{5.605 \cdot 10^{-4}}{10^{-6}} 10^{-6} = 0.5605 \cdot 10^{-3} \text{ (volts)}
\]

Case II: \( C \leq C_0 \) then Equation (10.37) can be simplified

\[
V \equiv \frac{5.605 \cdot 10^{-4}}{C_0 \left(1 - k_{31}^2\right)} S_i \text{ (volts)}
\]

then for every micro-strain,

\[
V = \frac{5.605 \cdot 10^{-4}}{3.304 \cdot 10^{-9}} 10^{-6} = 0.1696 \text{ (volts)}
\]

In conclusion of the two cases, a significant higher voltage can be obtained on a smaller external capacitor. In the applications, this indicates that a measuring device of very small input capacitance, normally this means high input impedance, is desirable.
10.10 **Electromechanical Response of Free PWAS**

This section addresses the behavior of a free piezoelectric wafer active sensor. The modeling of a free piezoelectric sensor is useful for (a) understanding the electromechanical coupling between the mechanical vibration response and the complex electrical response of the sensor; and (b) sensor screening and quality control prior to installation on the monitored structure.

Consider a piezoelectric wafer of length $l$, width $b$, and thickness $t$, undergoing longitudinal expansion, $u_1$, induced by the thickness polarization electric field, $E_3$. The electric field is produced by the application of a harmonic voltage $V(t) = \tilde{V}_0 e^{i\omega t}$ between the top and bottom surfaces (electrodes). The resulting electric field, $E_3 = \tilde{V}_0/\ell$, is assumed uniform over the piezoelectric wafer.

![Diagram of a piezoelectric active sensor and infinitesimal axial element](image)

**Figure 10.4** Schematic of a piezoelectric active sensor and infinitesimal axial element

Assume that the length, width, and thickness have widely separated values ($t<<b<<l$) such that the length, width, and thickness motions are practically uncoupled. The motion which is predominantly in the longitudinal direction, $x_1$ will be considered (1-D assumption). Since the electric field is uniform over the piezoelectric wafer, its derivative
is zero, i.e., \( \frac{\partial E_3}{\partial x_i} = 0 \). Since the voltage excitation is harmonic, the electric field has the expression \( E_3 = \hat{E}_3 e^{j\omega t} \), and the response is also harmonic, i.e., \( u = \hat{u} e^{j\omega t} \), where \( \hat{u}_i(x) \) is the \( x \)-dependant complex amplitude that incorporates any phase difference between the excitation and response. For compactness, use the notations \( \frac{\partial}{\partial x_i}(\cdot) = (\cdot)' \) and \( \frac{\partial}{\partial t}(\cdot) = (\cdot)'' \). Under the 1-D assumptions, the general constitutive equations reduce to the simpler expressions:

\[
S_i = s_{11}^E T_i + d_{31} E_3 \quad \text{(Strain)} \tag{10.38}
\]

\[
D_3 = d_{31} T_i + e_{33}^T E_3 \quad \text{(Electric displacement)} \tag{10.39}
\]

where \( S_1 \) is the strain, \( T_1 \) is the stress, \( D_3 \) is the electrical displacement (charge per unit area), \( s_{11}^E \) is the mechanical compliance at zero field, \( e_{33}^T \) is the dielectric constant at zero stress, and \( d_{31} \) is the induced strain coefficient, i.e., mechanical strain per unit electric field. Typical values of these constants are given in Table 10.2.

<table>
<thead>
<tr>
<th>Property</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Compliance</td>
<td>( s_{11}^E )</td>
<td>( 15.30 \times 10^{-12} ) Pa(^{-1} )</td>
</tr>
<tr>
<td>Dielectric constant</td>
<td>( e_{33}^T )</td>
<td>( 15.47 \times 10^9 ) F/m</td>
</tr>
<tr>
<td>Induced strain coefficient</td>
<td>( d_{31} )</td>
<td>( -175 \times 10^{-12} ) m/V</td>
</tr>
<tr>
<td>Coupling factor</td>
<td>( k_{31} )</td>
<td>0.36</td>
</tr>
<tr>
<td>Sound speed</td>
<td>( c )</td>
<td>2900 m/s</td>
</tr>
</tbody>
</table>

Recall Newton's law of motion and the strain-displacement relation:

\[
T_i' = \rho \ddot{u}_i \quad \text{(Newton's law of motion)} \tag{10.40}
\]
\[ S_1' = u_1' \quad \text{(Strain-displacement relation)} \] \hspace{1cm} (10.41)

Differentiate Equation (1) with respect to \( x \); since \( E_3 \) is constant (i.e. \( E'_3 = 0 \)), the strain rate becomes:

\[ S'_1 = s_{11}^E \cdot T'_1 \] \hspace{1cm} (10.42)

Substitution of Equations (3) and (4) into Equation (5) gives:

\[ u''_1 = s_{11}^E \rho \ddot{u}_1 \] \hspace{1cm} (10.43)

Introduce the notation

\[ c^2 = \frac{1}{\rho s_{11}^E} \quad \text{(Wave speed)} \] \hspace{1cm} (10.44)

representing the longitudinal wave speed of the material. Substitution of Equation (7) into Equation (6) yields the 1-D wave equation:

\[ \ddot{u}_1 = c^2 u''_1 \] \hspace{1cm} (10.45)

The general solution of the wave equation is:

\[ u_1(x,t) = \hat{u}(x)e^{i\omega t} \] \hspace{1cm} (10.46)

where

\[ \hat{u}_1(x) = C_1 \sin \gamma x + C_2 \cos \gamma x \] \hspace{1cm} (10.47)

and

\[ \gamma = \frac{\omega}{c} \] \hspace{1cm} (10.48)
The quantity $\gamma$ is the wave number, while the quantity $\lambda$ is the wavelength. The constants $C_1$ and $C_2$ are determined from the boundary conditions.

10.11 MECHANICAL RESPONSE

For a piezoelectric wafer of length $l$ with the origin at its center, the interval of interest is $x \in (-\frac{1}{2}l, \frac{1}{2}l)$. Stress-free boundary conditions imply $T_1(-\frac{1}{2}l) = T_1(+\frac{1}{2}l) = 0$.

Substitution into Equation (1) yields

$$S_1\left(-\frac{1}{2}l\right) = d_{31}E_3 \tag{10.49}$$

$$S_1\left(+\frac{1}{2}l\right) = d_{31}E_3 \tag{10.50}$$

Using the strain-displacement relation of Equation (4) gives

$$\ddot{u}_1\left(-\frac{1}{2}l\right) = \gamma \left(C_1 \cos \gamma \frac{1}{2}l + C_2 \sin \gamma \frac{1}{2}l\right) = d_{31}\ddot{E}_3 \tag{10.51}$$

$$\ddot{u}_1\left(+\frac{1}{2}l\right) = \gamma \left(C_1 \cos \gamma \frac{1}{2}l - C_2 \sin \gamma \frac{1}{2}l\right) = d_{31}\ddot{E}_3 \tag{10.52}$$

Addition of the two equations yields an equation in only $C_1$, i.e.,

$$\gamma C_1 \cos \gamma \frac{1}{2}l = d_{31}\ddot{E}_3 \tag{10.53}$$

Hence,

$$C_1 = \frac{d_{31}\ddot{E}_3}{\gamma \cos \frac{1}{2}\gamma l} \tag{10.54}$$

Subtraction of the two equations yields an equation in only $C_2$, i.e.,

$$\gamma C_2 \sin \gamma \frac{1}{2}l = 0 \tag{10.55}$$

Assuming $\sin \frac{1}{2}\gamma l \neq 0$, we get
\[ C_2 = 0 \]  

(10.56)

An alternative derivation of \( C_1 \) and \( C_2 \) can be obtained using Cramer’s rule on the algebraic system:

\[
\begin{align*}
C_1 \gamma \cos \frac{1}{2} \gamma l + C_2 \gamma \sin \frac{1}{2} \gamma l &= d_{31} \hat{E}_3 \\
C_1 \gamma \cos \frac{1}{2} \gamma l - C_2 \gamma \sin \frac{1}{2} \gamma l &= d_{31} \hat{E}_3
\end{align*}
\]  

(10.57)

Hence,

\[
C_1 = \frac{d_{31} \hat{E}_3 \gamma \sin \frac{1}{2} \gamma l}{\gamma \cos \frac{1}{2} \gamma l \gamma \sin \frac{1}{2} \gamma l - \gamma \cos \frac{1}{2} \gamma l \gamma \sin \frac{1}{2} \gamma l} = \frac{d_{31} \hat{E}_3 (-2 \gamma \sin \frac{1}{2} \gamma l)}{-\gamma^2 2 \sin \frac{1}{2} \gamma l \cos \frac{1}{2} \gamma l} = \frac{d_{31} \hat{E}_3}{\gamma \cos \frac{1}{2} \gamma l}
\]  

(10.58)

and

\[
C_2 = \frac{\gamma \cos \frac{1}{2} \gamma l \gamma \cos \frac{1}{2} \gamma l - \gamma \cos \frac{1}{2} \gamma l \gamma \sin \frac{1}{2} \gamma l}{\gamma \cos \frac{1}{2} \gamma l \gamma \cos \frac{1}{2} \gamma l - \gamma \cos \frac{1}{2} \gamma l \gamma \sin \frac{1}{2} \gamma l} = \frac{0}{-\gamma^2 2 \sin \frac{1}{2} \gamma l \cos \frac{1}{2} \gamma l} = 0
\]  

(10.59)

In this derivation, we assumed that the determinant appearing at the denominator is nonzero, i.e., \( \Delta \neq 0 \). Since \( \Delta = -\gamma^2 2 \sin \frac{1}{2} \gamma l \cos \frac{1}{2} \gamma l = \gamma^2 \sin \gamma l \), this condition implies that \( \sin \gamma l \neq 0 \).

Substitution of \( C_1 \) and \( C_2 \) into Equation (10) yields

\[
\hat{u}_i(x) = d_{31} \hat{E}_3 \frac{\sin \gamma x}{\gamma \cos \frac{1}{2} \gamma l}
\]  

(10.60)

Using Equation (4) we get the strain as:
\[ \hat{S}_s(x) = d_{31} \hat{E}_3 \frac{\cos \gamma x}{\cos \frac{1}{2} \gamma l} \quad (10.61) \]

10.12 Solution in Terms of the Induced Strain, \( S_{\text{ISA}} \), and Induced Displacement, \( u_{\text{ISA}} \)

Introduce the notations

\[ S_{\text{ISA}} = d_{31} \hat{E}_3 \quad \text{(Induced strain)} \quad (10.62) \]

\[ u_{\text{ISA}} = S_{\text{ISA}} l = (d_{31} \hat{E}_3) l \quad \text{(Induced displacement)} \quad (10.63) \]

where ISA signifies "induced strain actuation". Hence, Equations (23)-(24) can be written as

\[ \hat{u}(x) = \frac{1}{2} u_{\text{ISA}} \frac{\sin \gamma x}{\frac{1}{2} \gamma l \cos \frac{1}{2} \gamma l} \quad (10.64) \]

\[ \hat{S}(x) = S_{\text{ISA}} \frac{\cos \gamma x}{\cos \frac{1}{2} \gamma l} \quad (10.65) \]

Equations (27)-(28) show that, under dynamic conditions, the strain along the piezoelectric wafer is not constant. This is in contrast with the static case, for which the strain is uniform along the length of the actuator. The maximum strain amplitude is observed in the middle, and has the value

\[ S_{\text{max}} = \frac{S_{\text{ISA}}}{\cos \frac{1}{2} \gamma l} \quad (10.66) \]

Introducing the notation \( \phi = \frac{1}{2} \gamma l \), the displacement and strain equations can be rewritten as:

135
\[ \hat{u}(x) = \frac{1}{2} \frac{\mu_{ISA}}{\phi} \frac{\sin \gamma x}{\cos \phi} \]  
(10.67)

\[ \hat{S}_1(x) = S_{ISA} \frac{\cos \gamma x}{\cos \phi} \]  
(10.68)

### 10.13 Tip Strain and Displacement

At the wafer ends, \( x = \pm \frac{1}{2} l \), the tip strain and displacement are:

\[ \hat{u}(\pm \frac{1}{2} l) = \pm \frac{1}{2} \frac{\mu_{ISA}}{\gamma l} \sin \frac{1}{2} \gamma l \pm \frac{1}{2} \frac{\mu_{ISA}}{\gamma l} \tan \frac{1}{2} \gamma l \]  
(10.69)

\[ \hat{S}_1(\pm \frac{1}{2} \gamma l) = S_{ISA} \frac{\cos \frac{1}{2} \gamma l}{\cos \frac{1}{2} \gamma l} = S_{ISA} \]  
(10.70)

We notice that, at the tip, the strain takes exactly the value \( S_{ISA} \).

![Figure 10.5](image-url)

**Figure 10.5** Schematic of a 1-D piezoelectric active sensor under electric excitation

### 10.14 Electrical Response

Recall Equation (2) representing the electrical displacement:

\[ D_3 = d_3 r_1 + e_3^* E_3 \]  
(10.71)

Equation (1) yields the stress as function of strain and electric field, i.e.,
\[ T_i = \frac{1}{s_{11}} \left( S_1 - d_{31} E_3 \right) \]  \hspace{1cm} (10.72)

Hence, the electric displacement can be expressed as:

\[ D_3 = \frac{d_{31}}{s_{11}} \left( S_1 - d_{31} E_3 \right) + \varepsilon_{33} E_3 \]  \hspace{1cm} (10.73)

Upon substitution of the strain-displacement relation of Equation (4), we get:

\[ D_3 = \frac{d_{31}}{s_{11}} \cdot \mathbf{u}'_i - \frac{d_{31}^2}{s_{11}}\cdot E_3 + \varepsilon_{33} E_3 \]  \hspace{1cm} (10.74)

i.e.,

\[ D_3 = \varepsilon_{33} E_3 \left[ 1 - k_{31}^2 \left( 1 - \frac{\mathbf{u}_i'}{d_{31} E_3} \right) \right] \]  \hspace{1cm} (10.75)

where \( k_{31}^2 = \frac{d_{31}^2}{s_{11} \varepsilon_{33}} \) is the electromechanical coupling coefficient. Integration of

Equation (38) over the electrodes area \( A = bl \) yields the total charge:

\[ Q = \int_A D_3 dxdy = \int_{-l/2}^{l/2} \int_0^b D_3 dxdy = \varepsilon_{33} E_3 b \left[ l - k_{31}^2 \left( l - \frac{1}{d_{31} E_3} \cdot \mathbf{u}_i \right) \right] \]  \hspace{1cm} (10.76)

Assuming harmonic time dependence, \( Q = \hat{Q} e^{i\omega t} \), yields \( \hat{Q} \) as function of \( \mathbf{u}_i \) at both ends of the piezoelectric wafer:

\[ \hat{Q} = \varepsilon_{33} E_3 b \left[ l - k_{31}^2 \left( l - \frac{1}{d_{31} E_3} \cdot [\mathbf{u}_i(\frac{l}{2}) - \mathbf{u}_i(-\frac{l}{2})] \right) \right] \]  \hspace{1cm} (10.77)

Rearranging,
\[ \dot{Q} = \varepsilon_{33}^T b l \left[ 1 - k_{31}^2 + k_{31}^2 \left( \frac{\hat{u}_1(\frac{1}{2} l) - \hat{u}_1(-\frac{1}{2} l)}{d_{31} \hat{E}_3 \cdot l} \right) \right] \]  

(10.78)

Using the definitions \( C = \varepsilon_{33}^T \frac{A}{h} \), \( A = bl \), \( u_{ISA} = d_{31} \hat{E}_3 \cdot l \), \( \dot{V} = \frac{\hat{E}_3}{h} \), we obtain

\[ \dot{Q} = C \dot{V} \left[ 1 - k_{31}^2 + k_{31}^2 \left( \frac{\hat{u}_1(\frac{1}{2} l) - \hat{u}_1(-\frac{1}{2} l)}{u_{ISA}} \right) \right] \]

(10.79)

The electric current is obtained as the time derivative of the electric charge, i.e.,

\[ I = \dot{Q} = i \omega Q \]

(10.80)

Hence,

\[ \dot{i} = i \omega C \dot{V} \left[ 1 - k_{31}^2 + k_{31}^2 \left( \frac{\hat{u}_1(\frac{1}{2} l) - \hat{u}_1(-\frac{1}{2} l)}{u_{ISA}} \right) \right] \]

(10.81)

Alternatively, the electric current could have been obtained by performing first the derivative with respect to time, and then the integration with respect to space, i.e.,

\[ I = \int \frac{dD_3}{dt} dA = i \omega \int_D dA \]

(10.82)

The integral in Equation (45) is the same as the integral in Equation (39).

The admittance, \( Y \), is defined as the ratio between the current and voltage, i.e.,

\[ Y = \frac{\dot{i}}{\dot{V}} = i \omega C \left[ 1 - k_{31}^2 + k_{31}^2 \left( \frac{\hat{u}_1(\frac{1}{2} l) - \hat{u}_1(-\frac{1}{2} l)}{u_{ISA}} \right) \right] \]

(10.83)

Recall Equation (27) giving the displacement solution:
\[ \hat{u}_1(x) = \frac{d_{ij}}{\gamma} \frac{E_j}{\sin \gamma x \cos \frac{1}{2} \gamma l} \]

Hence,

\[
\frac{\hat{u}_1(\frac{1}{2} l) - \hat{u}_1(-\frac{1}{2} l)}{u_{ISA}} = 2 \frac{\sin \frac{1}{2} \gamma l - (-\sin \frac{1}{2} \gamma l)}{\frac{1}{2} \gamma l \cos \frac{1}{2} \gamma l} = 2 \frac{\sin \frac{1}{2} \gamma l}{\frac{1}{2} \gamma l \cos \frac{1}{2} \gamma l} = \frac{\tan \frac{1}{2} \gamma l}{\frac{1}{2} \gamma l} \tag{10.84}
\]

where the notation \( \phi = \frac{1}{2} \gamma l \) was invoked. Substitution in Equation (46) yields

\[
Y = i\omega C \left[ 1 - k_{31}^2 \left( 1 - \frac{\tan \frac{1}{2} \gamma l}{\frac{1}{2} \gamma l} \right) \right] \tag{10.85}
\]

This result agrees with Ikeda (1996). However, it may be more convenient, at times, to write it as:

\[
Y = i\omega C \left[ 1 - k_{31}^2 \left( 1 - \frac{1}{\frac{1}{2} \gamma l \cdot \cot \frac{1}{2} \gamma l} \right) \right] \tag{10.86}
\]

Note that the admittance is purely imaginary, and consists of the capacitive admittance, \( i\omega C \), modified by the effect of piezoelectric coupling between mechanical and electrical variables. This effect is apparent in the term containing the electromechanical coupling coefficient, \( k_{31}^2 \). The impedance, \( Z \), is obtained as the ratio between the voltage and current, i.e.,

\[
Z = \frac{\dot{V}}{\dot{I}} = Y^{-1} \tag{10.87}
\]

Hence,

\[
Z = \frac{1}{i\omega C} \left[ 1 - k_{31}^2 \left( 1 - \frac{\tan \frac{1}{2} \gamma l}{\frac{1}{2} \gamma l} \right) \right]^{-1} \tag{10.88}
\]
Recalling the notation $\phi = \frac{1}{2} \gamma l$, we can write

$$Y = i\omega C \left[ 1 - k^2_{ji} \left( 1 - \frac{\tan \phi}{\phi} \right) \right]$$

(10.89)

$$Z = \frac{1}{i\omega C} \left[ 1 - k^2_{ji} \left( 1 - \frac{\tan \phi}{\phi} \right) \right]^{-1}$$

(10.90)
CHAPTER 11  PLATES UNDER PWAS EXCITATION

Assume a distributed shear force applied to the surface of the plate on the area covered by the PWAS. Consider a plate with thickness $2d$, and a PWAS with width $2a$ applied to the top surface of the plate. The PWAS is under a harmonic loading, and shear stress is introduced to the surface of the plate which can be expressed as $\tau = \tau_0(x)e^{iat}$ (Figure 1). $\tau_0(x)$ is the distribution function of the shear stress.

![Figure 11.1](image)

**Figure 11.1**  Plate of thickness $2d$, with a PWAS of width $2a$, under harmonic loading on the top surface.

Assume displacement as function of scalar potential $\Phi$ and vector potential $\mathbf{H}$, such that

$$\mathbf{u} = \nabla \Phi + \nabla \times \mathbf{H}, \quad \nabla \cdot \mathbf{H} = 0$$  \hspace{1cm} (11.1)

where $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$, and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the unit vectors in $x$, $y$, and $z$ directions respectively. The first term in $\mathbf{u}$ is the gradient of the scalar potential $\Phi$, while the second term is the curl of the zero-divergence vector potential $\mathbf{H}$. Expand Equation (4.1) to get:

$$\mathbf{u} = \frac{\partial \Phi}{\partial x}\mathbf{i} + \frac{\partial \Phi}{\partial y}\mathbf{j} + \frac{\partial \Phi}{\partial z}\mathbf{k} + \left(\frac{\partial H_y}{\partial z} - \frac{\partial H_z}{\partial y}\right)\mathbf{i} + \left(\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z}\right)\mathbf{j} + \left(\frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x}\right)\mathbf{k}$$  \hspace{1cm} (11.2)
For the case in discussion, plane strain in \( z \) is assumed, which gives \( u_z = \partial / \partial z = 0 \). Hence

\[
H_x = H_y = 0 , \quad \text{and the non-zero displacements are}
\]

\[
\begin{align*}
    u_x &= \frac{\partial \Phi}{\partial x} + \frac{\partial H_z}{\partial y} \\
    u_y &= \frac{\partial \Phi}{\partial y} - \frac{\partial H_z}{\partial x}
\end{align*}
\]  \hspace{1cm} (11.3)

while the operator \( \nabla \) is simplified as \( \nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} \). The wave equations in terms of \( \Phi \) and \( H_z \) are written as

\[
\begin{align*}
    \nabla^2 \Phi &= \frac{1}{c_p^2} \frac{\partial^2 \Phi}{\partial t^2} \\
    \nabla^2 H_z &= \frac{1}{c_s^2} \frac{\partial^2 H_z}{\partial t^2}
\end{align*}
\]  \hspace{1cm} (11.4)

where \( c_p \) and \( c_s \) are the principal (pressure) and secondary (shear) wave velocities.

Consider the harmonic solution of the wave equations in the form

\[
\begin{align*}
    \Phi &= f e^{-i \omega t} \\
    H_z &= h_z e^{-i \omega t}
\end{align*}
\]  \hspace{1cm} (11.5)

where \( t^2 = -1 \). Substitution of Equation (4.5) into Equation (4.4) gives:

\[
\begin{align*}
    \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} &= -\frac{\omega^2}{c_p^2} f \\
    \frac{\partial^2 h_z}{\partial x^2} + \frac{\partial^2 h_z}{\partial y^2} &= -\frac{\omega^2}{c_s^2} h_z
\end{align*}
\]  \hspace{1cm} (11.6)

For simplicity, the time harmonic term \( e^{i \omega t} \) is omitted.
Taking Fourier transform in terms of $x$ yields:

\[ -\xi^2 \tilde{f} + \frac{d^2 \tilde{f}}{dy^2} = -\frac{\omega^2}{c_p^2} \tilde{f} \]
\[ -\xi^2 \tilde{h}_s + \frac{\partial^2 \tilde{h}_s}{\partial y^2} = -\frac{\omega^2}{c_s^2} \tilde{h}_s \]  

(11.7)

Rearranging Equation (11.7) and letting $\alpha^2 = \frac{\omega^2}{c_p^2} - \xi^2$, $\beta^2 = \frac{\omega^2}{c_s^2} - \xi^2$ gives

\[ \frac{d^2 \tilde{f}}{dy^2} + \alpha^2 \tilde{f} = 0 \]
\[ \frac{\partial^2 \tilde{h}_s}{\partial y^2} + \beta^2 \tilde{h}_s = 0 \]  

(11.8)

Hence,

\[ \tilde{f} = A_s \sin \alpha y + A_c \cos \alpha y \]
\[ \tilde{h}_s = B_s \sin \beta y + B_c \cos \beta y \]  

(11.9)

The sine terms are anti-symmetric in $y$, while the cosine terms are symmetric in $y$.

Consider the particle motion in a plate; the corresponding symmetric and anti-symmetric motions are illustrated in Figure 11.2.

---

4 The Fourier transform of $f(x)$ is defined as $\tilde{f}(\xi) = F(f(x)) = \int_{-\infty}^{\infty} f(x) e^{i\xi x} dx$, and the inverse Fourier transform is defined as $f(x) = F^{-1}(\tilde{f}(\xi)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\xi) e^{i\xi x} d\xi$. Fourier transform of derivatives are given by $F\left(f^{(n)}(x)\right) = i^n \xi^n F\left(f(x)\right)$, such that

\[ F\left(f''(x)\right) = i\xi^2 F\left(f(x)\right) = i\xi \tilde{f}(\xi), \text{ and } F\left(f''(x)\right) = i^2 \xi^2 F\left(f(x)\right) = -\xi^2 \tilde{f}(\xi) \]

(Spiegel, 1968, pp174, 176)
Figure 11.2  Symmetric and anti-symmetric particle motion across the plate thickness.

Note that for symmetric motion with respect to the mid plane, $u_x$, follows the cosine behavior, while $u_y$ follows the sine behavior. Similarly, for anti-symmetric motion, $u_x$ follows the sine behavior, while $u_y$ follows to cosine behavior. Hence, Equation (4.8) is regrouped to the following two sets of functions,

$$\tilde{f}_s = A_2 \cos \alpha y \quad \text{symmetric motion}$$

$$\tilde{h}_{ss} = B_1 \sin \beta y$$

(11.10)

and

$$\tilde{f}_a = A_1 \sin \alpha y \quad \text{anti-symmetric motion}$$

$$\tilde{h}_{sa} = B_2 \cos \beta y$$

(11.11)

To utilize the free-surface condition, stress distribution needs to be calculated. The two stress components in concern are $\tau_{yx}$ and $\tau_{xy}$. In terms of potentials $\Phi$ and $H_x$, these are

$$\tau_{yx} = \mu \left( 2 \frac{\partial^2 \Phi}{\partial x \partial y} + \frac{\partial^2 H_x}{\partial y^2} - \frac{\partial^2 H_x}{\partial x^2} \right)$$

(11.12)

and

$$\tau_{xy} = \left( \lambda + 2\mu \right) \left( \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right) - 2\mu \left( \frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_x}{\partial x \partial y} \right)$$

(11.13)
Taking Fourier transform in terms of $x$, we get the transformed stresses

$$
\tilde{\tau}_{xx} = \mu \left( 2i\xi \frac{df}{dy} + \frac{d^2\tilde{h}_z}{dy^2} + \xi^2 \tilde{h}_z \right)
$$

(11.14)

$$
\tilde{\tau}_{yy} = (\lambda + 2\mu) \left( -\xi^2 \tilde{f} + \frac{d^2\tilde{f}}{dy^2} \right) - 2\mu \left( -\xi^2 \tilde{f} + i\xi \frac{d\tilde{h}_z}{dy} \right)
$$

(11.15)

Without losing generality, let's consider the symmetric motion. The derivative terms in Equations (11.14) and (11.15) are

$$
\frac{df}{dy} = -\alpha A_y \sin \alpha y
$$

$$
\frac{d^2f}{dy^2} = -\alpha^2 A_y \cos \alpha y
$$

(11.16)

$$
\frac{d\tilde{h}_z}{dy} = \beta B_1 \cos \beta y
$$

$$
\frac{d^2\tilde{h}_z}{dy^2} = -\beta^2 B_1 \sin \beta y
$$

Substituting Equations (11.16) into Equations (11.14) yields

$$
\tilde{\tau}_{xx} = \mu \left( 2i\xi \frac{df}{dy} + \frac{d^2\tilde{h}_z}{dy^2} + \xi^2 \tilde{h}_z \right)
$$

$$
= \mu \left( -2i\xi \alpha A_y \sin \alpha y - \beta^2 B_1 \sin \beta y + \xi^2 B_1 \sin \beta y \right)
$$

(11.17)

Substituting Equation (11.16) into Equations (11.15), and rearranging the terms, gives
\[
\tilde{\tau}_{yy} = (\lambda + 2\mu) \left( -\xi^2 \bar{f} + \frac{d^2 \bar{f}}{dy^2} \right) - 2\mu \left( -\xi^2 \bar{f} + i\xi \frac{d\bar{h}}{dy} \right)
\]
\[
= (\lambda + 2\mu) \left( -\xi^2 A_2 \cos \alpha y - \alpha^2 A_2 \cos \alpha y \right) - 2\mu \left( -\xi^2 A_2 \cos \alpha y + i\xi \beta B_1 \cos \beta y \right) - \lambda \xi^2 \bar{A}_2 \cos \alpha y - 2\mu \xi \beta B_1 \cos \beta y
\]
\[
= \mu \left( \xi^2 - \beta^2 \right) A_2 \cos \alpha y - 2\mu \xi \beta B_1 \cos \beta y
\]

where

\[
-\lambda \xi^2 - (\lambda + 2\mu) \alpha^2 = -\lambda \xi^2 - (\lambda + 2\mu) \left( \frac{\omega^2}{c_p^2} - \xi^2 \right)
\]
\[
= -\lambda \xi^2 - (\lambda + 2\mu) \left( \frac{\omega^2}{(\lambda + 2\mu) \rho} \xi^2 \right) = 2\mu \xi^2 - \frac{\omega^2}{1 \rho}
\]
\[
= \mu \left( 2\xi^2 - \frac{\omega^2}{\mu \rho} \right) = \mu \left( 2\xi^2 - \frac{\omega^2}{c_s^2} \right) = \mu \left( \xi^2 - \frac{\omega^2}{c_s^2} - \xi^2 \right)
\]
\[
= \mu \left( \xi^2 - \beta^2 \right)
\]

Similarly procedure can be applied to obtain the transformed shear stress for anti-symmetric motion; the final results are

\[
\tilde{\tau}_{yy}^{\tilde{S}} = \mu \left[ -2i\xi A_2 \sin \alpha y + \left( \xi^2 - \beta^2 \right) B_1 \sin \beta y \right]
\]

\[
\tilde{\tau}_{yy}^{\tilde{S}} = \mu \left( \xi^2 - \beta^2 \right) A_2 \cos \alpha y - 2\mu \xi \beta B_1 \cos \beta y
\]

symmetric motion

(11.19)

and

\[
\tilde{\tau}_{yy}^{\tilde{A}} = \mu \left[ -2i\xi A_2 \cos \alpha y + \left( \xi^2 - \beta^2 \right) B_1 \cos \beta y \right]
\]

\[
\tilde{\tau}_{yy}^{\tilde{A}} = \mu \left( \xi^2 - \beta^2 \right) A_2 \sin \alpha y - 2\mu \xi \beta B_1 \sin \beta y
\]

anti-symmetric motion

(11.20)

To incorporate the symmetric and anti-symmetric motion, a pair of self-equilibrating shear stresses is introduced to the lower surface. The two shear stresses have same amplitude, \(\tau / 2\), one pointing to the +x direction, and the other pointing to the -x direction.
The shear stress on the upper surface is divided into two equal parts, and then grouped with the two newly introduced shear stresses to form symmetric and anti-symmetric shear stress pairs (Figure 11.3).

\[ \tau_s \big|_{y=-d} = \frac{1}{2} \tau_0(x) e^{i\alpha} \]
\[ \tau_s \big|_{y=d} = \frac{1}{2} \tau_0(x) e^{-i\alpha} \]

(a)
\[ \tau_a \big|_{y=-d} = \frac{1}{2} \tau_0(x) e^{i\beta} \]
\[ \tau_a \big|_{y=d} = -\frac{1}{2} \tau_0(x) e^{-i\beta} \]

(b)

Figure 11.3 Plate of thickness 2d, with a PWAS of width 2a: (a) under symmetric loading and (b) under anti-symmetric loading.

Apply the boundary condition to Equations (11.19) and (11.20). For the symmetric motion,

\[ \frac{1}{\mu} \left[ \begin{array}{l} \tau_{sx} \big|_{y=-d} \\ \tau_{sy} \big|_{y=-d} \end{array} \right] = \left( \xi^2 - \beta^2 \right) A_2 \sin \alpha d + \left( \xi^2 - \beta^2 \right) B_1 \sin \beta d = \frac{1}{\mu} \frac{\tau_0}{2} \]

\[ \frac{1}{\mu} \left[ \begin{array}{l} \tau_{sx} \big|_{y=d} \\ \tau_{sy} \big|_{y=d} \end{array} \right] = \left( \xi^2 - \beta^2 \right) A_2 \cos \alpha d - 2i \xi \beta B_1 \cos \beta d = 0 \]  

(11.21)

In matrix form,

\[ \begin{bmatrix} -2i \xi \alpha \sin \alpha d & \left( \xi^2 - \beta^2 \right) \sin \beta d \\ \left( \xi^2 - \beta^2 \right) \cos \alpha d & -2i \xi \beta \cos \beta d \end{bmatrix} \begin{bmatrix} A_2 \\ B_1 \end{bmatrix} = \begin{bmatrix} \frac{\tau_0}{2\mu} \\ 0 \end{bmatrix} \]

(11.22)

Let
\[ D_3 = \begin{vmatrix} -2i\xi \alpha \sin \alpha d & \left(\xi^2 - \beta^2\right) \sin \beta d \\ \left(\xi^2 - \beta^2\right) \cos \alpha d & -2i\xi \beta \cos \beta d \end{vmatrix} = -4\alpha \beta \xi^2 \sin \alpha d \cos \beta d - \left(\xi^2 - \beta^2\right)^2 \cos \alpha d \sin \beta d \] (11.23)

\[ N_{A_1} = \begin{vmatrix} \tilde{r}_0 & \left(\xi^2 - \beta^2\right) \sin \beta d \\ 0 & -2i\xi \beta \cos \beta d \end{vmatrix} = -\frac{\tilde{r}_0}{\mu} i\xi \beta \cos \beta d \] (11.24)

\[ N_{B_1} = \begin{vmatrix} -2i\xi \alpha \sin \alpha d & \tilde{r}_0 \\ \left(\xi^2 - \beta^2\right) \cos \alpha d & 0 \end{vmatrix} = -\frac{\tilde{r}_0}{2\mu} \left(\xi^2 - \beta^2\right) \cos \alpha d \] (11.25)

Hence, the two unknown coefficients can be solved,

\[ A_2 = \frac{N_{A_1}}{D_3} = \frac{\begin{vmatrix} \tilde{r}_0 & \left(\xi^2 - \beta^2\right) \sin \beta d \\ 0 & -2i\xi \beta \cos \beta d \end{vmatrix}}{D_3} = -\frac{\tilde{r}_0}{\mu} \frac{i\xi \beta \cos \beta d}{D_3} \] (11.26)

\[ B_1 = \frac{N_{B_1}}{D_3} = \frac{\begin{vmatrix} -2i\xi \alpha \sin \alpha d & \tilde{r}_0 \\ \left(\xi^2 - \beta^2\right) \cos \alpha d & 0 \end{vmatrix}}{D_3} = -\frac{\tilde{r}_0}{2\mu} \frac{\left(\xi^2 - \beta^2\right) \cos \alpha d}{D_3} \]

Similarly, coefficients for anti-symmetric motion are obtained from the system:

\[ \frac{1}{\mu} \tilde{r}_\alpha^{\mu} \mid_{y=0} = -2i\xi \alpha A_1 \cos \alpha d + \left(\xi^2 - \beta^2\right) B_2 \cos \beta d = \frac{1}{\mu} \tilde{r}_0 \frac{2}{2} \] (11.27)

\[ \frac{1}{\mu} \tilde{r}_\alpha^{\mu} \mid_{y=0} = \left(\xi^2 - \beta^2\right) A_1 \sin \alpha d - 2i\xi \beta B_2 \sin \beta d = 0 \]

In matrix form,
\[
\begin{bmatrix}
-2i\xi\alpha \cos \alpha d & (\xi^2 - \beta^2) \cos \beta d \\
(\xi^2 - \beta^2) \sin \alpha d & -2i\xi\beta \sin \beta d 
\end{bmatrix}
\begin{bmatrix}
A_1 \\
B_2 
\end{bmatrix}
= \begin{bmatrix}
\frac{\bar{\tau}_0}{2\mu} \\
0 
\end{bmatrix}
\] (11.28)

Let

\[
D_A = \begin{vmatrix}
-2i\xi\alpha \cos \alpha d & (\xi^2 - \beta^2) \cos \beta d \\
(\xi^2 - \beta^2) \sin \alpha d & -2i\xi\beta \sin \beta d 
\end{vmatrix} = -4\xi^2 \alpha \beta \cos \alpha d \sin \beta d - (\xi^2 - \beta^2)^2 \sin \alpha d \cos \beta d
\] (11.29)

\[
N_A = \begin{vmatrix}
\frac{\bar{\tau}_0}{2\mu} & (\xi^2 - \beta^2) \cos \beta d \\
0 & -2i\xi\beta \sin \beta d 
\end{vmatrix} = -\frac{\bar{\tau}_0}{\mu} i\xi\beta \sin \beta d
\] (11.30)

\[
N_B = \begin{vmatrix}
-2i\xi\alpha \cos \alpha d & \frac{\bar{\tau}_0}{2\mu} \\
(\xi^2 - \beta^2) \sin \alpha d & 0 
\end{vmatrix} = -\frac{\bar{\tau}_0}{2\mu} (\xi^2 - \beta^2) \sin \alpha d
\] (11.31)

Solve for

\[
A_1 = \frac{N_A}{D_A} = -\frac{\bar{\tau}_0}{\mu} \frac{i\xi\beta \sin \beta d}{D_A}
\]

\[
B_2 = \frac{N_B}{D_A} = -\frac{\bar{\tau}_0}{2\mu} \frac{(\xi^2 - \beta^2) \sin \alpha d}{D_A}
\] (11.32)

It is noticed that by letting \(D_S = 0\) and \(D_A = 0\), Rayleigh-Lamb frequency equations for symmetric and anti-symmetric motion are obtained. For example,

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\[ D_S \begin{vmatrix} -2i\xi\alpha \sin \alpha d & (\xi^2 - \beta^2) \sin \beta d \\ (\xi^2 - \beta^2) \cos \alpha d & -2i\xi\beta \cos \beta d \end{vmatrix} = 0 \]

\[ \Rightarrow 4\xi^2 a \beta \sin \alpha d \cos \beta d = -\left(\xi^2 - \beta^2\right)^2 \cos \alpha d \sin \beta d \]

\[ \Rightarrow -\frac{4a\beta\xi^2}{\left(\xi^2 - \beta^2\right)^2} = \frac{\tan \beta d}{\tan \alpha d} \]

which is Rayleigh-Lamb frequency equation for the propagation of symmetric waves in a plate. Also, letting \( D_S = 0 \) and \( D_A = 0 \) corresponds to the homogeneous solutions of Equations (11.22) and (11.28) respectively. These are the solutions for symmetric and anti-symmetric Lamb waves in a load-free plate with free surfaces. From Rayleigh-Lamb frequency equations, the wave number \( \xi \) can be resolved. Using the simple relation \( \xi = \frac{\omega}{c} \), phase velocity of the propagating wave is obtained.

Once the coefficients are solved, the Fourier transformed potential functions can be derived by substituting Equations (11.26) and (11.32) into Equation (4.8)

\[ \tilde{f} = -\frac{\tilde{\tau}_0}{\mu} i\xi \beta \left( \cos \beta d \cos \alpha y + \frac{\sin \beta d \sin \alpha y}{D_S + D_A} \right) \]

\[ \tilde{h}_z = -\frac{\tilde{\tau}_0}{2\mu} (\xi^2 - \beta^2) \left( \frac{\cos \alpha d \sin \beta y}{D_S} + \frac{\sin \alpha d \cos \beta y}{D_A} \right) \] (11.33)

Applying Fourier transform to the displacements in Equation (4.3) yields

\[ \tilde{u}_x = i\xi \tilde{f} + \frac{d\tilde{h}_z}{dy} = \frac{\tilde{\tau}_0}{2\mu} \left( \frac{N_{sS}}{D_S} + \frac{N_{sA}}{D_A} \right) \]

\[ \tilde{u}_y = \frac{df}{dy} - i\xi \tilde{h}_z = i\frac{\tilde{\tau}_0}{2\mu} \left( \frac{N_{sS}}{D_S} - \frac{N_{sA}}{D_A} \right) \] (11.34)

Where
\[\begin{align*}
N_{sx} &= 2\xi^2 \beta \cos \beta d \cos \alpha y - \beta (\xi^2 - \beta^2) \cos \alpha d \cos \beta y \\
N_{sA} &= 2\xi^2 \beta \sin \beta d \sin \alpha y + \beta (\xi^2 - \beta^2) \sin \alpha d \sin \beta y \\
N_{yS} &= 2\xi \alpha \beta \cos \beta d \sin \alpha y + \xi (\xi^2 - \beta^2) \cos \alpha d \sin \beta y \\
N_{yA} &= 2\xi \alpha \beta \sin \beta d \cos \alpha y - \xi (\xi^2 - \beta^2) \sin \alpha d \cos \beta y
\end{align*}\] (11.35)

By inverse Fourier transform, the displacements can be derived as

\[
\begin{align*}
u_x &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\tau}_0 \frac{N_{sS} + N_{sA}}{D_s + D_A} \frac{1}{2\mu} e^{i\xi x} d\xi \\
u_y &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{i}{2\mu} e^{i\xi x} \frac{N_{yS} - N_{yA}}{D_s - D_A} \frac{1}{2\mu} d\xi
\end{align*}\] (11.36)

Equation (11.36) can be calculated using Cauchy's residue theorem:

\[
\begin{align*}
u_x &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\tau}_0 \frac{N_{sS} + N_{sA}}{D_s + D_A} \frac{1}{2\mu} e^{i\xi x} d\xi = \frac{1}{2\pi i} \frac{1}{2\mu} \left( \sum_{\xi^S} \text{Res} (\xi^S) + \sum_{\xi^A} \text{Res} (\xi^A) \right) \\
&= \frac{i}{2\mu} \left( \sum_{\xi^S} \text{Res} (\xi^S) + \sum_{\xi^A} \text{Res} (\xi^A) \right)
\end{align*}\] (36a)

\[
\begin{align*}
u_y &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{i}{2\mu} e^{i\xi x} \frac{N_{yS} - N_{yA}}{D_s - D_A} \frac{1}{2\mu} d\xi = \frac{1}{2\pi i} \frac{i}{2\mu} \left( \sum_{\xi^S} \text{Res} (\xi^S) - \sum_{\xi^A} \text{Res} (\xi^A) \right) \\
&= -\frac{1}{2\mu} \left( \sum_{\xi^S} \text{Res} (\xi^S) - \sum_{\xi^A} \text{Res} (\xi^A) \right)
\end{align*}\] (36b)

Where \(\sum_{\xi^S} \text{Res} (\xi^S)\) and \(\sum_{\xi^A} \text{Res} (\xi^A)\) are the terms of the residues of the function when wave number \(\xi\) is the root of \(D_s = 0\) or \(D_A = 0\), respectively.

In practice, however, strain measurements are of more interest. In PWAS applications, for example, it's the strain on the upper and/or lower surface(s) that will be detected.

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For the case in discussion, we only concern with $\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$. Apply Fourier transform to obtain:

$$\tilde{\varepsilon}_{xx} = i\xi \tilde{u}_x$$

(11.37)

Using Equation (11.34) and (11.35), the Fourier transformed strain is

$$\tilde{\varepsilon}_{xx} = i\xi \tilde{u}_x = i\xi \frac{\tilde{\tau}_0}{2\mu} \left( \frac{N_{ss}}{D_s} + \frac{N_{SA}}{D_A} \right)$$

(11.38)

By applying inverse Fourier transform, the strain distribution can be obtained

$$\varepsilon_{xx} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{\varepsilon}_{xx} e^{i\xi x} d\xi = \frac{1}{2\pi} \int_{-\infty}^{+\infty} i\xi \frac{\tilde{\tau}_0}{2\mu} \left( \frac{N_{ss}}{D_s} + \frac{N_{SA}}{D_A} \right) e^{i\xi x} d\xi$$

$$= \frac{1}{2\pi} \frac{i}{2\mu} \int_{-\infty}^{+\infty} \xi \tilde{\tau}_0 \left( \frac{N_{ss}}{D_s} + \frac{N_{SA}}{D_A} \right) e^{i\xi x} d\xi$$

(11.39)

Equation (11.39) can be calculated using Cauchy’s residue theorem:

$$\varepsilon_{xx} = \frac{1}{2\pi} \frac{i}{2\mu} \int_{-\infty}^{+\infty} \xi \tilde{\tau}_0 \left( \frac{N_{ss}}{D_s} + \frac{N_{SA}}{D_A} \right) e^{i\xi x} d\xi$$

$$= 2\pi i \frac{1}{2\mu} \left( \sum_{\xi} \text{Res} \left( \xi^S \right) + \sum_{\xi} \text{Res} \left( \xi^A \right) \right) = -\frac{1}{2\mu} \left( \sum_{\xi} \text{Res} \left( \xi^S \right) + \sum_{\xi} \text{Res} \left( \xi^A \right) \right)$$

(11.40)

Where $\sum \text{Res} \left( \xi^S \right)$ and $\sum \text{Res} \left( \xi^A \right)$ are the terms of the residues of the function when wave number $\xi$ is the root of $D_s = 0$ or $D_A = 0$, respectively. According to the Rayleigh-Lamb frequency equations Equations (11.23) and (11.29), roots exist in pairs. At a given frequency, $\omega$, the roots are $\pm \xi^S_0, \pm \xi^A_0, \pm \xi^S_1, \pm \xi^A_1 \ldots, \pm \xi^S_n, \pm \xi^A_n$ ($n_S$ and $n_A$ are the highest mode numbers available at the given value of $\omega$).
Figure 11.4 Dispersion curves for a free isotropic aluminum plate (c_v=6.35 mm/μs, c_f=3.13 mm/μs, c/f=2.0288). (Rose, 1999, pp109)

For simplicity, let's consider the situation at low frequency, where only S_0 and A_0 modes exist (Figure 11.4). The integration contour is chosen to include the positive wave numbers (ξ^S_0, ξ^A_0), and exclude the negative wave numbers (-ξ^S_0, -ξ^A_0), as shown in Figure 11.5. For higher frequencies, where more modes are present, the same principle will apply.

For \( f(z) = \frac{A(z)}{B(z)} \) with simple pole at \( z_0 \) (i.e. \( B(z_0) = 0 \)), the residue at \( z = z_0 \) is

\[
\text{Res}(z_0) = \frac{A(z_0)}{B'(z_0)} \quad \text{(Hildebrand, 1976)}.
\]

Hence, Equation (11.40) becomes:

\[
\varepsilon_{xx} = -\frac{1}{2\mu} \left( \bar{\tau}_0 \sum_{S} \xi^S N_{xx} \left( \xi^S \right) \frac{N_{xx} \left( \xi^A \right)}{D_A' \left( \xi^A \right)} + \bar{\tau}_0 \sum_{A} \xi^A N_{xx} \left( \xi^A \right) \frac{N_{xx} \left( \xi^A \right)}{D_A' \left( \xi^A \right)} \right)
\]

(40a)
Figure 11.5 Contour for evaluating the surface strain under S0 and A0 mode. Residues at positive wave numbers are included, and excluded for the negative wave numbers.

Now, let's look at a particular case where ideal bonding is assumed and the surface stress can be modeled using the pin-force model. In this model, a pair of opposing forces of magnitude $F = \tau_o a$ are applied to the edges of the PWAS. Hence the shear stress is

$$\tau = \tau_o a \left[ \delta(x-a) - \delta(x+a) \right].$$

Figure 11.6 shows the configuration and the distribution of the force.

![Configuration and force distribution](image)

Figure 11.6 (a) Plate of thickness $2d$, with a PWAS of width $2a$, under pin-force loading on the top surface. (b) The stress distribution on the top surface

Apply Fourier transform; we get
\[ \tilde{\tau}(\tilde{\xi}) = F \left[ \delta(x-a) - \delta(x+a) \right] \tau_0 a \\
= \left[ F \left( \delta(x-a) \right) - F \left( \delta(x+a) \right) \right] \tau_0 a \\
= \left( e^{-i\xi a} - e^{-i\xi(-a)} \right) \tau_0 a = (-2i \sin \xi a) \tau_0 a \] (11.41)

On the surface, \( y = d \), we have the following values,

\[ N_{sS} \big|_{y=d} = \beta \left( \xi^2 + \beta^2 \right) \cos \alpha d \cos \beta d \]
\[ N_{sA} \big|_{y=d} = \beta \left( \xi^2 + \beta^2 \right) \sin \alpha d \sin \beta d \] (11.42)

Substitute Equations (11.41) and (11.42) into the strain Equation (11.38)

\[ \bar{\varepsilon}_{xx} = i\xi \frac{1}{2\mu} \left( N_{sS} D_S + N_{sA} D_A \right) \]
\[ = i\xi \frac{1}{2\mu} \left( -2i \sin \xi a \right) \tau_0 a \left( N_{sS} D_S + N_{sA} D_A \right) \]
\[ = \frac{\tau_0 a}{\mu} \xi \sin \xi a \left( N_{sS} D_S + N_{sA} D_A \right) \] (11.43)

Applying inverse Fourier transform and using the residue method, the strain can be obtained as:

\[ \varepsilon_{xx} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \frac{\tau_0 a}{\mu} \xi \sin \xi a \left( N_{sS} D_S + N_{sA} D_A \right) e^{i\xi} \right] d\xi \]
\[ = \frac{1}{2\pi i} \left( \sum_{e} \xi \sin \xi a \frac{N_{sS}}{D_S} e^{i\xi} + \sum_{e} \xi \sin \xi a \frac{N_{sA}}{D_A} e^{i\xi} \right) \]

i.e.

\[ \varepsilon_{xx} = \frac{\tau_0 a}{\mu} \left[ \sum_{e} \xi \sin \xi a \frac{N_{sS}}{D_S} e^{i\xi} + \sum_{e} \xi \sin \xi a \frac{N_{sA}}{D_A} e^{i\xi} \right] \] (11.44)
Using the Lamb wave phase velocity calculated in Chapter 4, the wave numbers can be obtained. With these wave numbers, the $S_0$ and $A_0$ mode Lamb wave strain can be calculated. Figure 11.7(a) shows a plot of the normalized strain amplitude of the two modes. The parameters are $a=7$ mm, $2d=1.6$ mm, and the material is aluminum. It is noticed that the $S_0$ mode has its peak value at around 300 kHz, while the $A_0$ mode is almost zero at this frequency. So this is a frequency point where “pure” $S_0$ mode Lamb wave can be excited.

![Normalized strain curve](image1)

(a) $S_0$ mode and $A_0$ mode normalized strain

![Response curve](image2)

(b) Experiment

**Figure 11.7** $S_0$ mode Lamb wave is selectively excited at 300 kHz. (a) Normalized strain amplitude of $S_0$ and $A_0$ mode Lamb waves, theoretical value; (b) PWAS signal amplitude of $S_0$ and $A_0$ mode Lamb waves, experimental result.
An experiment was setup to verify this “mode tuning” effect. A 1.6 mm thick, 2024-aluminum alloy beam (914 x 12.7 x 1.6 mm) was instrumented with two 7 x 7 mm PZT wafer active sensors (APC 850). One sensor is located at the center of the beam. The distance between the two PWAS is 200 mm. The sensors were connected with thin insulated wires to a HP33120A arbitrary signal generator. Smoothed tone-burst excitation with a 10 Hz repetition rate was used. The signal was sent to the PWAS at the center, and the response of the other PWAS was collected by a Tektronix TDS210 digital oscilloscope. From 10 to 600 kHz, the excitation signal frequency was tested with a 10 kHz step. The shapes of the received signal were used to identify the wave modes. The $S_0$ mode is not dispersive, so the signal is similar to the excitation signal and not distorted. The $A_0$ mode is very dispersive, which is represented by wide spread wave packets. The amplitude of the received signal were recorded and plotted versus the frequency in Figure 11.7 (b). From the comparison of the theoretical and experimental results, the “mode tuning” effects is clearly demonstrated.
PART III APPLICATIONS OF PWAS TO ULTRASONIC NDE
CHAPTER 12  STATE OF THE ART IN PWAS ULTRASONICS AND NDE

Pierce et. al. (1997), Culshaw, Pierce, Staszekski (1998), and Gachagan (1999) introduced a Structurally Integrated System for the comprehensive evaluation of Composites (SISCO). It was an acoustic/ultrasonic based structural monitoring system for composite structures. Conventional ultrasonic transducers were coupled to the sample via perspex wedges to generate Lamb waves, and the responses were monitored using optical fiber sensors. They also studied a low profile acoustic source (LPAS) in this system. The LPAS used a 1-3 connectivity piezo-composite layer (Figure 12.1a) as the active phase and two flexible printed circuit boards (PCB) as the upper and lower electrodes. The PCB was designed with a interdigital (ID) electrode pattern. (Figure 12.1b)

![Diagram](image)

**Figure 12.1**  Construction of 1-3 connectivity piezo-composite (Pierce, S. G., et. al., 1997) and the interdigital pattern on the electrode of LPAS transducer (Gachagan, A., et. al., 1999)

The spacing of the ID pattern was designed to be one half of the desired Lamb wavelength and the fingers of the ID pattern were driven differentially. The overall
thickness of the LPAS was approximately 0.7 mm. A comparison of the traditional ultrasonic transducers and the LPAS is shown in Figure 12.2.

Figure 12.2  Conventional wedge ultrasonic sources based on 1:3 connectivity composite actuators and new low profile geometries for future system integration (Culshaw, B. Pierce, S. G., Staszekski, W. J., 1998)

Figure 12.3  $S_0$ Lamb wave reflection from holes: upper trace shows defect-free sample, lower trace shows 10 mm hole. (Culshaw, B. Pierce, S. G., Staszekski, W. J., 1998)

The experiments showed that although the signal levels from LPAS were lower than that from conventional ultrasonic transducers with a perspex wedge, LPAS are capable to
generate Lamb waves in composite plates. A sample signal collected by the optical fiber sensor is showing in Figure 12.3.

Lamb waves generated from LPAS propagate along the direction of LPAS. The Lamb wave “beam” divergence is reported to be 15.5 degrees. The author suggested using a curved ID pattern to broaden the beam, which is more proper for a condition monitoring system.

![Figure 12.4](image.png)

**Figure 12.4** Multiple 5 mm diameter, 0.1mm thick PZT discs were used as transducers to detect defects in a 16 ply C/epoxy plate. (Lemistre et al. 2000)

Lemistre, Osmont, and Balageas (2000) presented an active health system based on wavelet transform analysis. Multiple 5 mm diameter 0.1 mm thick PZT discs were used as transducers. These transducers were used as actuators and receivers alternatively. They were mounted on the surface of a 700x700x2 mm 16 ply C/epoxy composite plate. During tests, a burst of 10-count, 365 kHz sinusoidal signal was applied to one of the
transducers; hence Lamb waves were generated in the plate. Other transducers then picked up the signal, and were collected by a digitizer. The signal was then processed using discrete wavelet transform to isolate the S\textsubscript{0} and A\textsubscript{0} Lamb modes, then the Time of Flight (TOF) was obtain from the isolated signals. Since the Lamb wave speed is known, once the TOF is obtained, the distance between the sensor and the defects can be calculated. Using the distance to multiple sensors, the location of the defects can be determined.

![Diagram](image)

**Figure 12.5** Schematic of a piezo integrated health monitoring system based on modal acoustic emission. (Osmont, et al., 2000)

Osmont et al. (2000) presented damage and damaging impact monitoring system using PZT sensors. Multiple 5 mm diameter 0.1 mm thick PZT discs were used as sensors (Figure 12.5). These PZT discs were mounted (a) inside and (b) on the surface of carbon-epoxy plates. Each plate is impacted using a weight drop machine. The impact energy varies from 2 J to 6 J for 16 ply coupons, and 4 J to 8 J for 32 ply coupons. Acoustic Emission (AE) during the impact propagates throughout the plates and registered by the
PZT sensors. High Frequency Root Mean Square (HF-RMS) value of the acoustic signal was used as damage severity factor.

![Figure 12.6](image)

Two PZT patches (20x5 mm) were bonded close to the end of a narrow carbon-epoxy composite beam to detect the controlled delamination at the center of the beam. One of the PZT patch is used as actuator and the other as sensor. (Diaz Valdes, and Soutis, 2000)

Diaz Valdes and Soutis (2000) demonstrated an application of PZT patch in detecting delamination in a Carbon-epoxy composite beam. Two 20x5 mm PZT patches were made from commercial brass backed piezoceramic resonators. They were bonded on the surface of the beam close to one end. (Figure 12.6). The Carbon-epoxy composite beam is 629x25 mm. A PC with a data acquisition card (DAQ) was used to generate excitation signal and collect the response from the PZT transducers. 5.5-count, 15kHz sinusoidal pulse were used as the excitation signal. One of the two PZT patches was used as transducer, and the other was used as sensor. The baseline signal was first collected from the pristine beam and stored. A blade was then used to create a small delamination at the center of the beam. The blade was forced into the beam to increase the area-of-delamination ($A_d$). The PZT transducer was excited and the response on the sensor was collected and stored. These signals then were compared with the baseline signal to show the presence of the delamination. Figure 12.7 shows the signals at different area-of-delamination (Column 1) and their difference from the baseline signal (Column 2). The effects of the delamination were pointed out in column 2 signals. The delamination is indicated by the presence of new wave packets, and the size of delamination can be
estimated by the amplitude of the wave packets. In the sensor received signals, the second largest group of signals is from the far end of the beam.

![Graphs showing amplitude over time](image)

Figure 12.7 Column 1: PZT sensor received signal when the area-of-delamination is (a) 0 (baseline signal), (b) 22 mm², (c) 47 mm², and (d) 220 mm². Column 2: Arithmetic difference between the signals in column 1 and the baseline signal. The reflections from delamination is denoted by the pointer. (Diaz Valdes and Soutis, 2000)

Lin and Yuan (2001a) presented an approach to detect and image multiple damages in a plate-like structure. A migration technique used in geophysical exploration was adopted to interpret the backscattering wave field and to image the flaws in the structure. The
finite difference method was used to simulate the reflection waves and in implementing the prestack migration. An analytical solution based on Mindlin plate theory was derived to verify the accuracy of the numerical algorithm. Numerical results showed that multiple damages can be successfully detected. This approach is proposed to be applied to a linear array of piezo actuators/sensors.

Lin and Yuan (2001b) studied the modeling of diagnostic transient wave in an integrated piezoelectric sensor/actuator plate. PZT ceramic disks were mounted on the surface of an aluminum plate acting as both actuators and sensors to generate and collect $A_0$ mode Lamb waves. The propagating waves were modeled with Mindlin plate theory. The interaction between the actuator and the plate was modeled using classical lamination theory. The sensor acts as a capacitor that converts the sensed strain change into a voltage response. An analytical expression for the sensor output voltage in terms of the given input excitation signal is derived, and agree with the experimental results.

![Diagram](image)

**Figure 12.8** Two PZT ceramic disks (PKI-402, 6.36 mm diameter, 0.25 mm thick) were used in pair to generate $A_0$ mode Lamb wave in a 1020x810 mm Al-6061 aluminum plate. Another PZT ceramic disk was used as sensor to collect the wave signal. (Lin and Yuan, 2001b)

In the experiments, PZT ceramic disks PKI-402 (6.36 mm diameter, 0.25 mm thick) from Piezo Kinetics Inc. were used. The 1020x810 mm plate was made of Al-6061 aluminum alloy. Hewlett Packard 33120A waveform generator was used to generate the Hanning
windowed 5-count 100 kHz sinusoidal excitation signal. Krohn-Hite 7602 wideband power amplifier was used to amplify the signal. National Instruments 5911 data acquisition board was used to collect the signal from the PZT sensor.

As shown in Figure 12.8, two PZT disks were aligned and mounted on the upper and lower surfaces of the plate. Their polarization direction was aligned with z-axis of the plate. When the excitation voltage $V_{in}$ was applied to both actuators, equal traction with opposite direction will be applied on the two surfaces of the plate. The resulting bending moment on the plate along the circumference of the actuator is uniform and flexural waves are therefore generated.

![Graphs showing sensor output with different actuator-sensor distances]

**Figure 12.9** Sensor output with different actuator-sensor distances ($f_c = 100$ kHz): (a) $R=150$ mm, (b) $R=200$ mm, and (c) $R=250$ mm. (Lin and Yuan, 2001)

Figure 12.9 shows the comparison of the analytical prediction and experimental data of the sensor signal voltage. The results show that the experimental data agree very well
with calculated data for both the amplitude and phase of the response. It also shows that the received signal retains the shape of the excitation signal. This means by using the Hanning window, excitation signal has a narrow frequency band; hence the dispersive effects of the Lamb wave were suppressed.

Figure 12.10  Four piezoelectric disks (APC 850, 6.35 mm diameter, 0.25 mm thick) were bonded on the surface of a square aluminum alloy plate (508x508x1.02 mm). Time-reversal method were used to focus the excitation signal to a simulated scatter. (Wang, Rose, and Chang, 2003)

Wang, Rose, and Chang (2003) introduced a computerized time-reversal method for structural health monitoring. Four piezoelectric disks (APC 850, 6.35 mm diameter, 0.25 mm thick) were bonded on the surface of a square aluminum alloy plate (508x508x1.02 mm). For the time-reversal method, tone burst of center frequency of 50 kHz was used. During the test one of these transducers (A) was used as actuator, and the other three were
used as sensors ($B$, $C$, and $D$). The received signals are the sensors are shown in Figure 12.11. The time of flight (TOF) of the first main pulse was denoted by $T_B$, $T_C$, and $T_D$, respectively.

![Graphs showing the time of flight for transducers B, C, and D.]

Figure 12.11 Scattered responses at (a) transducer $B$, (b) transducer $C$, and (c) transducer $D$. With proper time-delay, excitations from transducers $B$, $C$, and $D$ are focused at the location of transducer $A$, a much stronger signal is detected (d) (Wang, Rose, and Chang, 2003)

A maximum TOF is defined as

$$T_{\text{max}} = \max(T_i), \quad (i = B, C, D) \quad (12.1)$$

Then the necessary time-delay for each transducer is given by, in consistent with time-reversal concept,

$$\delta T_i = T_{\text{max}} - T_i, \quad (i = B, C, D) \quad (12.2)$$
The excitation signal then was sent to each of the transducers (B, C, and D) deferred by the corresponding time-delay. A much stronger signal is received at transducer A (Figure 12.11d).

PWAS as the active component of these innovative ultrasonic NDE methods was theoretically studied in the previous two Parts of this dissertation. Part III will cover the experimental aspects of the research. Excitation modulation and signal processing techniques were studied. A phased array algorithm was also introduced, which can be used scan large area of a structure. Experimental result is provided to demonstrate the capability of PWAS in generating and detecting Lamb waves in metallic thin plates.
CHAPTER 13  TONE BURST EXCITATION SIGNALS

Lamb waves are dispersive. After traveling a long distance, wave packets of different frequencies will separate from each other and distort the signal, and make it difficult for the analysis. To minimize this effect, an excitation signal of minimal frequency components is preferable. A commonly used excitation signal is tone burst signal (Lemistre et al. 2000, Diaz Valdes and Soutis, 2000, Lin and Yuan, 2001a, 2001b), which is a wave train consisting several cycles of the same frequency. To reduce the frequency component, a smoothing window can be used to modulate the tone burst signal.

A 3-count Hanning windowed tone-burst signal was used in this study (Figure 13.1c). The original tone-burst signal is defined as:

$$g(t) = \begin{cases} \sin(\omega t) & 0 \leq t \leq T_c \\ 0 & t > T_c \end{cases}$$  \hspace{1cm} (13.1)$$

where $\omega = 2\pi f$, \quad $T = \frac{1}{f}$, \quad $T_c = N_c T$, \quad $N_c$ is the number of counts of the waves.

The Hanning window was defined by

$$h(t) = 0.5 \left[ 1 - \cos \left( \frac{1}{N_c} \omega t \right) \right] \quad 0 \leq t \leq T_c \hspace{1cm} (13.2)$$

The excitation is the convolution of the tone-burst and the Hanning window, $g(t) * h(t)$. By using the convolution theorem for Fourier transforms, it can be shown that the Fourier
The transform of the excitation signal is the product of the Fourier transform of the two functions,

\[ F\{g * h\} = F\{g\}F\{h\} \]  \hspace{1cm} (13.3)

Figure 13.1  Priniciples of smooth tone burst signal: (a) 3-count 300kHz tone burst; (b) Hanning window; (c) smoothed tone burst

Fourier transform demonstrated that the use of Hanning window eliminated the side lobes. (Figure 13.2) It is noticed that the Fourier transform of the original tone burst signal has a main frequency peak and several small side band peaks. (Figure 13.2 a2). The frequency component of this signal is spread in a large range. The Fourier transform of the Hanning windowed tone burst (Figure 13.2 b2) shows a wider main frequency peak, and there is almost no side band peak. In this case, the frequency components is concentrated around the main peak, hence reduce the frequency spread range.
Figure 13.2  Smoothing effects of Hanning windowing is to eliminate the side bands of the signal frequency spectrum. Figures shown are signals and their Fourier transform (a1) 3-count 300 kHz tone burst; (b1) smoothed tone burst; (a2) Fourier transform of 3-count tone burst shows significant side bands; (b2) Fourier transform of smoothed tone burst has almost no side bands but a wider main peak.

In the experiments, this Hanning windowed tone burst excitation signal were transmitted and received by PWAS transducers. The signals were received with little distortion. The signals will be demonstrated in Chapters 14 and 15.
CHAPTER 14  SIGNAL PROCESSING TECHNIQUE

Signal processing is important in the process of extracting useful information from the experimental data. The first step usually is noise reduction. Because the PWAS is very sensitive to vibrations, noise is inevitable. Especially when used in the field, there are lots of vibration sources. Fortunately the frequency used is very high (in hundreds kilohertz to megahertz range). This is significantly beyond the frequency of usual noise sources. Hence, the noise that appears in the raw signal from the PWAS is random noise. The sample averaging method is implemented to remove this type of noise. Once noise is removed, the wave packages are very clear. Time of flight (TOF), signal amplitude and other important information can be visually recognized. Due to the dispersive nature of Lamb waves, wave packages are distorted, and the exact TOF could be difficult to recognize. Hence more signal processing steps are taken to assist the analysis. In the experiments, several signal processing techniques were used. Cross correlation was used to extract the location of the original wave packages. Hilbert transform was used to obtain the envelope of the wave package. Then the TOF of the wave packages was easily detected with a simple peak detection algorithm. These signal processing techniques are discussed in this chapter.
14.1 SAMPLE AVERAGING

Figure 14.1 Comparison of wave signals (a) before and (b) after 64-sample averaging.

Sample averaging is useful when random noise is present in the signal. By averaging multiple samples the accuracy of the measurements can be improved. The presumption of averaging is that noise and measurement errors are random, and therefore, by the central
limit theorem, (Miller, 1999) the error will have a normal (Gaussian) distribution. The central limit theorem states that if $\bar{X}$ is the mean of a sample of size $n$ taken from a population having the mean $\mu$ and the finite variance $\sigma^2$, then $Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}$ is a random variable whose distribution function approaches that of the standard normal distributions as $n \to \infty$.

In the wave propagation experiment, random noise is significant in Figure 14.1a shows the signal collected directly from the PWAS. After 64-sample averaging, the noise is greatly reduced, and the signal shows clearly the wave pattern (Figure 14.1b).

It also is noticed that the standard deviation of $Z$ is proportional to $1/\sqrt{n}$. Therefore, the more samples are taken in the average, the smaller the standard deviation is. However, this process is limited by the hardware's sampling rate, the memory buffer size, and processing speed. The data acquisition hardware in the experiments is a Tektronix TDS-210 digital oscilloscope. This oscilloscope can take average on 4, 16, 64, or 128 samples. Consider the repetition rate of the pulse signal is 64 Hz, the 64-sample averaging is selected. In our experiments, the sample averaging is done in the data acquisition hardware. Although the sample averaging could be also done in the data processing software, this would not be advantageous due to the slow speed of the data communication interface between the digital oscilloscope and the PC.

14.2 CROSS-CORRELATION

Cross correlation is used to detect similarities in two signals. The cross correlation $R_{xy}(t)$ of two signals $x(t)$ and $y(t)$ is defined by
\[ R_{xy}(t) = \int_{-\infty}^{+\infty} x(\tau) y(\tau + t) d\tau \]  

(14.1)

For example, Figure 14.2 shows a 3-count sinusoidal signal \( x(t) = \sin(2\pi \cdot t) \) generated at sensor X starting at \( t_x = 0 \), and vibrating for 3 cycles. This signal is then received at \( t = 6 \) by sensor Y, as signal \( y(t) \). The function \( R_{xy}(t) \), shown in Figure 14.2, presents a maximum at \( t_y = 6 \).

The process of calculating the cross correlation can be explained as following: moving the input signal along the time axis by a small time step and check the similarity of the two signals. When the input signal is moved to \( t = t_y = 6 \), the two signals match each other exactly, and the cross correlation function \( R_{xy}(t) \) reaches maximum.

\[ \begin{array}{c}
\text{excitation signal} \quad x(t) \\
\text{sensor signal} \quad y(t) \\
\text{cross correlation signal} \quad R_{xy}(t) \\
\end{array} \]

\[ \text{time (s)} \]

Figure 14.2 Cross correlation \( R_{xy} \) shows the similarity of two signals \( x(t) \) and \( y(t) \). It has a maximum value when the two signals match.

Figure 14.3 illustrates the cross-correlation principle when windowed tone-burst wave signals are used. Assume the time-of-flight (TOF) of the wave signal is \( t_y = 20 \) microseconds. By calculating the cross correlation of the excitation signal and the sensor signal, the time of arrival is clearly obtained at the maximum of the cross correlation signal.
excitation signal \( x(t) \)
sensor signal \( y(t) \)
cross correlation signal \( R_{xy}(t) \)

Figure 14.3  Estimate time of arrival using cross correlation.

Using the cross correlation between the transmitted signal and the reception signals also reduces the noise. Figure 14.4 shows an example containing experimental data. The excitation signal was a Hanning windowed 3-count 300 kHz tone-burst signal. The received signal carries significant noise, which hampers the detection of the time of flight of the wave packet. The cross correlation signal shown in the last row of Figure 9 displays significantly reduced noise and clearly shows the moment of arrival.

Figure 14.4  Cross correlation eliminates noise and shows time of arrival clearly.
14.3 **Hilbert Transform and Envelope of the Signal**

The envelope is a curve or surface that is tangent to every one of a family of curves or surfaces. Envelope extracts the amplitude of a periodic signal, and can help to simplify the process of detecting the time of arrival for the wave packets. The envelope of the signal is obtained by applying Hilbert transform to the cross correlation signal. The Hilbert transform of signal $x(t)$ is defined as

$$H(x(t)) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{x(\tau)}{t-\tau} d\tau.$$  \hspace{1cm} (14.2)

One can use the Hilbert transform to construct a complex signal:

$$\tilde{x}(t) = \tilde{x}_{\text{Re}}(t) + i \cdot \tilde{x}_{\text{Im}}(t)$$  \hspace{1cm} (14.3)

where

$$\tilde{x}_{\text{Re}}(t) = x(t)$$
$$\tilde{x}_{\text{Im}}(t) = H(x(t))$$

The real part, $\tilde{x}_{\text{Re}}(t)$, is the original data. The imaginary part $\tilde{x}_{\text{Im}}(t)$ is the Hilbert transform. The imaginary part is a version of the original real sequence with a 90° phase shift. Thus, the Hilbert transformed signal has the same amplitude and frequency content as the original real signal and includes phase information that depends on the phase of the original signal. The magnitude of each complex value $\tilde{x}(t)$ is the amplitude of the original signal. Thus, the magnitude of $\tilde{x}(t)$ is the envelope of the original signal. By observing the envelope signal, the wave packages can be easily recognized.
Using Hilbert transform to construct the envelop of a 3-count, Hanning windowed sinusoidal signal.

Figure 14.5 demonstrates the construction of the envelope. A 3-count Hanning windowed sinusoidal signal is generated to simulate the signal. The Hilbert transform is applied to this signal. The resulting Hilbert transform signal was used with the original signal to build the complex sequence $\tilde{x}(t)$. The envelope of the original signal then was created by calculating the magnitude of $\tilde{x}(t)$. It is shown clearly that the envelope extracts the shape of the wave packet, therefore retaining the amplitude which is of concern, and eliminating the phase of the signal. With the envelope signal, a simple peak detection algorithm can recognize the TOF easily.

Figure 14.6 Envelope of a signal represents the amplitude of the signal.

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Figure 14.6 shows extracting the envelope of the cross correlation signal of experimental data. The envelope simplifies the signal by extracting the signal profile, so that the maximum peaks of the cross correlation and the time of arrival can be determined by the local maximum of the envelop signal, which is much easier than from the original signal.
CHAPTER 15  PULSE-ECHO EXPERIMENTS

Figure 15.1  Plate, 1.6mm thick, 2024 Aluminum alloy

A 1.6 mm thick, 2024-aluminum alloy plate (914 x 504 x 1.6 mm) was instrumented with an array of eleven 7-mm x 7-mm PZT wafer active sensors. The (x, y) locations of the 11 active sensors are illustrated in Figure 15.1. The sensors were connected with thin insulated wires to a 16-channels signal bus that ended into two 8-pin connectors (Figure 15.2). An HP33120A arbitrary signal generator was used to generate a 300 kHz smoothed tone-burst excitation with a 10 Hz repetition rate. The signal was sent to active sensor #11. Under the tone burst excitation, the active sensor #11 generated a package of elastic waves that spread out into the entire plate according to a circular wave-front pattern. A Tektronix TDS210 two-channel digital oscilloscope, synchronized with the signal
generator, was used to collect the response signals from the active sensors. The two oscilloscope channels were digitally switched to one of the remaining 10 active sensors using a digitally controlled switching unit. A data acquisition program was developed using LabView software to control the signal switch and record the data from the digital oscilloscope. A Motorola MC68HC11 microcontroller was tested as an embedded stand-alone controlling option.
Figure 15.2  Experimental setup for rectangular plate wave propagation experiment: (a) overall view showing the plate, active sensors, and instrumentation; (b) detail of the microcontroller and switch box.

Figure 15.3 shows the signals received at the active sensors #1 through #10. In each signal, one notices a number of wave packs. The first of these packs corresponds to the wave received directly from the transmitter active sensor, #11. The subsequent wave
packs correspond to wave reflected from the boundaries. The time of flight (TOF) of each wave pack is consistent with the traveled distance.

![Graph showing waveforms for different sensors](image)

**Figure 15.3** Reception signals received at active sensors 1 through 10.
15.1 **GROUP VELOCITY ESTIMATION**

These raw signals of Figure 15.3 were processed using a narrow-band signal correlation algorithm followed by an envelope detection method. As a result, the exact TOF for each wave packet were precisely identified. To perform a group velocity estimation, the TOF of the first wave packets in each signal were plotted against the geometric distance between the receiving active sensor and the transmitting active sensor (Figure 15.4). It can be appreciated from Figure 15.4 that a perfect straight line (99.99% $R^2$ correlation) was obtained. The slope of this line is the group velocity. Its value is $c_g = 5.446$ km/s. For the 1.6-mm aluminum alloy used in this experiment, the theoretical $S_0$-mode speed at 300 kHz is $c_{S0} = 5.440$ km/s, as described in Chapter 4. Since, at this low frequency, the $S_0$ mode has negligible dispersion, the group velocity and the wave speed have practically the same values. Hence, we compared the experimentally determined value of 5.446 km/s
with the theoretical value of 5.440 km/s. The speed detection accuracy (0.1%) is remarkable. Based on the wave speed, we conclude that the first wave packets are $S_0$ Lamb waves.

15.2 REIFICATIONS ANALYSIS

The next step in the analysis was to understand the reflection patterns, represented by the subsequent wave packs appearing in each signal. These packs, appearing after the first pack, represent waves that are reflected from the edges of the plate. Understanding the wave reflection patterns is essential for establishing the methodology for implementing the pulse-echo method for damage detection. In the analysis, a connection between the TOF of each wave pack and the distance traveled (wave path length) were established. The TOF determination was immediate, but the determination of the actual traveled distance was more involved. Since elastic waves propagate in a circular wave front through the 2-D plate, they reflect at all the edges and continue to travel until fully attenuated. AutoCAD drawing software was used to assist the analysis and calculate the actual path length of the reflected waves. The drawing in Figure 15.5 shows the plate and its mirror reflections with respect to the 4 edges and 4 corners. Thus, an image containing nine adjacent plates was obtained. To identify which reflection path corresponds to a particular wave pack, we took its TOF and multiplied it by the $S_0$ Lamb wave group velocity to get a first estimate of the path length. For example, Figure 15.5 shows that for the reflected reception at sensor #6, the estimated distance is 645 mm. Then, a circle, centered at the transmitter sensor (#11) and with its radius equal to the estimated distance, was drawn. The circle intersects with, or very close to, one of the reflected images of the reception sensor (#6). In this way, we identified which of the reflected images of the
reception sensor #6 as the actual image to be processed. Next, the true distance (path
length) between the identified sensor image and the transmitter sensor was determined
using the distance feature of the AutoCAD software. In Figure 15.5, this true distance is
651.22 mm. In this way, the path lengths for the reflection wave packs of all sensors were
determined, and a TOF versus path length plot could be created (Figure 15.6). With the
exception of one outlier, the TOF vs. path length plot of Figure 27 shows good linearity
and thus proves the consistency of the reflection analysis.

![Diagram](image)

**Figure 15.5** Method for finding the distance traveled by the reflected waves. The actual plate
(shaded) is surrounded by eight mirror images to assist in the calculation of the
reflection path length.
Figure 15.6 Correlations between path length and time of flight for the wave reflection signals (2nd wave pack) captured by active sensors 1 through 10.

15.3 Pulse Echo Analysis

In our plate experiments, we were also able to perform pulse-echo analysis. After the initial excitation signal, the transmitter active sensor (#11) was also able to capture subsequent signals representing waves reflected by the plate boundaries (Figure 15.7). These subsequent wave packs were recorded for evaluating the pulse-echo method.

Figure 28a shows the sensor #11 signal. This signal contains the excitation signal (initial bang), and a number of wave packs received in the pulse-echo mode. The wave generated by the initial bang undergoes multiple reflections from the plate edges, as shown in Figure 15.7b. It should be noted that the path lengths for reflections R₁ and R₂ are very close. Hence, the echoes for these two reflections were superpose on the pulse-echo signal of Figure 15.7a. It is also important to notice that reflection R₄ has two possible paths, R₄a and R₄b. Both paths have the same length. Hence, the echoes corresponding to
these two reflection paths arrive simultaneously and form a single echo signal on Figure 15.7a, with roughly double the intensity of the adjacent signals. Figure 28c shows the TOF of the echo wave packages plotted against their path lengths. The straight line fit has a very good correlation ($R^2 = 99.99\%$). The corresponding wave speed is 5.389 km/s, i.e., within 1\% of the theoretical value of 5.440 km/s.
Figure 15.7  Pulse echo method applied to active sensor #11: (a) the excitation signal and the echo signals on active sensor 11; (b) schematic of the wave paths for each wave pack; (c) correlation between path length and time of flight.

15.4 MAIN FINDINGS OF THE PLATE EMBEDDED ULTRASONICS INVESTIGATION

The main findings of investigating embedded ultrasonics on a 2-D plate structure are:

Embedded ultrasonics experiments were successfully performed on a thin-gage rectangular plate specimen equipped with piezoelectric active sensors placed in an array pattern.

1. Though the PZT wafer active sensors had square shape, the waves were found to propagate in a circular wave front over the entire plate surface.

2. Though excitation was possible at various frequencies, it was found that the 300-kHz axial wave gave a more clear and distinct excitation and reception of wave packs than other waves and other frequencies.
3. The wave signals obtained in the 2-D plate are clean. This is attributed to the lack of detrimental interference from the sides of the specimen.

4. The wave speed was successfully determined and the theoretical value was recovered with remarkable accuracy (0.1%) for $S_0$ mode Lamb wave (not dispersive).

5. Due to the low attenuation of these waves, multiple reflections were present and showed in the captured signal as distinct wave packs. These multiple reflections were analyzed and successfully interpreted.

6. The use of the pulse-echo method with the embedded PZT wafer active sensor was successfully demonstrated. This opened the way for developing an embedded ultrasonics damage detection methodology.
CHAPTER 16  EMBEDDED ULTRASONIC STRUCTURE RADAR

The principle of operation of the embedded ultrasonic structural radar (EUSR) is derived, from two general principles (1) The principle of guided Lamb wave generation with piezoelectric wafer active sensors (PWAS); (2) The principles of conventional phased-array radar.

The guided Lamb waves generated by PWAS have the important property that they stay confined inside the walls of a thin-wall structure, and hence can travel over large distances. In addition, the guided wave can also travel inside curved walls, which makes them ideal for applications in the ultrasonic inspection of aircraft, missiles, pressure vessel, oil tanks, pipelines, etc. Lamb waves can exist in a number of dispersive modes. However, through smoothed tone-burst excitation and frequency tuning, it is possible to confine the excitation to a particular Lamb wave mode, of carrier frequency $F_c$, wave speed $c$, and wave length $\lambda = c/F_c$. Hence, the smoothed tone-burst signal generated by one PWAS is of the form:

$$s_r(t) = s_0(t) \cos 2\pi F_c t, \quad 0 < t < t_p$$  \hspace{1cm} (16.1)

where $s_0(t)$ is a short-duration smoothing window that is applied to the carrier signal of frequency $F_c$ between 0 and $t_p$. 

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The principles of conventional phased-array radar applied to PWAS-generated guided waves assumes a uniform linear array of $M$ sensors (PWAS), with each PWAS acting as a pointwise omni-directional transmitter and receiver. The PWAS in the array are spaced at the distance $d$, which is assumed much smaller than the distance $r$ to a generic, far-distance point, P. Since $d \ll r$, the rays joining the sensors with the point P can be assimilated with a parallel fascicle, of azimuth $\phi$ (Figure 16.1).
Figure 16.2  The basis of pulse-echo method: (a) transmitted smooth-windowed tone-burst of duration $t_p$; (b) received signal to be analyzed for the duration $t_0$, starting at $t_p$, in order to identify the time of flight delay, $\tau$ (after Silvia, 1987).

Because of the array spacing, the distance between one PWAS and the generic point P will be different from the distance between another PWAS and P. For the $m$-th PWAS, the distance will be shorted by $m(d\cos \phi)$. If all the PWAS are fired simultaneously, the signal from the $m$-th PWAS will arrive at P quicker by $\Delta_m(\phi) = m(d\cos \phi)/c$. If the PWAS are not fired simultaneously, but with some individual delays, $\delta_m$, $m = 0,1,\ldots,M-1$, then the total signal received at point P will be:

$$s_p(t) = \frac{1}{r} \sum_{m=0}^{M-1} s_1 \left( t - \frac{r}{c} + \Delta_m(\phi) - \delta_m \right)$$  \hspace{1cm} (16.2)

where $1/r$ represent the decrease in the wave amplitude due to the omni-directional 2-D radiation, and $r/c$ is the delay due to the travel distance between the reference PWAS ($m = 0$) and the point P. (wave-energy conservation, i.e., no dissipation, is assumed.)
Transmitter beamforming

Beamforming at angle $\phi_0$ with an array of $M$ omni-directional sensors is based on the principles of constructive interference in the fascicle of parallel rays emanating from the array. The simplest way of achieving constructive interferences, is to have $\delta_m = m\Delta(\phi)$, such that Equation (16.2) becomes

$$s_p(t) = M \cdot \frac{1}{r} s_r \left( t - \frac{r}{c} \right)$$

(16.3)

i.e. an $M$ times increase in the signal strength with respect to a simple sensor. This leads directly to the beamforming principle, i.e., if $\delta_m = m\frac{d}{c} \cos(\phi_0)$, and since

$$\Delta_m = m\frac{d}{c} \cos(\phi)$$

then constructive interference (beamforming) takes place when $\cos(\phi) = \cos(\phi_0)$, i.e. at angles $\phi = \phi_0$ and $\phi = -\phi_0$. Thus, the forming of a beam at angles...
\( \phi_0 \) and \(-\phi_0\) is achieved through delays in the firing of the sensors in the array. Figure 16.3 shows the beam forming pattern for \( \phi_0 = 53\) deg.

**Receiver beamforming**

The receiver beamforming principles are reciprocal to those of the transmitter beamforming. If the point P is an omni-directional source at azimuth \( \phi_0 \), then the signals received at the \( m \)-th sensor will arrive quicker by \( m \Delta_0(\phi) = m(d \cos \phi_0)/c \). Hence, we can synchronize the signals received at all the sensors by delaying them by

\[
\delta_m(\phi_0) = m \frac{d}{c} \cos(\phi_0)
\]

**Phased-array pulse-echo**

Assume that a target exists at azimuth \( \phi_0 \) and distance \( R \). The transmitter beamformer is sweeping the azimuth in increasing angles \( \phi \) and receives an echo when \( \phi = \phi_0 \). The echo will be received on all sensors, but the signals will not be synchronized. To synchronize the sensors signals, the delays \( \delta_m(\phi_0) = m \frac{d}{c} \cos(\phi_0) \) need to be applied. The process is as follows. The signal send by the transmitter beamformer is an \( M \) times boost or the original signal:

\[
s_p(t) = \frac{M}{R} s_\tau \left( t - \frac{2R}{c} \right)
\]  \hspace{1cm} (16.4)

At the target, the signal is backscattered with a backscatter coefficient, \( A \). Hence, the signal received at each sensor will be \( \frac{A \cdot M}{R^2} s_\tau \left( t - \frac{2R}{c} + \Delta_m(\phi) \right) \). The receiver beamformer assembles the signals from all the sensors with the appropriate time delays, i.e.,
\[ s_R(t) = \frac{A \cdot M}{R^2} \sum_{m=0}^{M-1} s_T \left( t - \frac{2R}{c} + \Delta_m(\phi) - \delta_m \right) \]  \hspace{1cm} (16.5)

Constructive interference between the received signals is achieved when
\[ \delta_m = m \frac{d}{c} \cos(\phi_0). \]
Thus, the assembled receive signal will be again boosted \( M \) times, with respect to the individual sensors, i.e.,
\[ s_R(t) = \frac{A \cdot M^2}{R^2} \sum_{m=0}^{M-1} s_T \left( t - \frac{2R}{c} \right) \]
\hspace{1cm} (16.6)

The time delay between the receive signal, \( s_R(t) \) and the transmit signal \( s_T(t) \), is
\[ \tau = \frac{2R}{c} \]
\hspace{1cm} (16.7)

Measurement of the time delay \( \tau \) observed in \( s_R(t) \) allows one to calculate the target range,
\[ R = c \tau/2. \]

Table 16.1 \( M \times M \) matrix of signal primitives generated in a round-robin phase-array activation of the active-sensor array

<table>
<thead>
<tr>
<th>Firing pattern (symbols designated the transmitters that are activated)</th>
<th>Synthetic beamforming response</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_0 )</td>
<td>( T_1 )</td>
</tr>
<tr>
<td>( R_0 )</td>
<td>( p_{0,0}(t) )</td>
</tr>
<tr>
<td>( R_1 )</td>
<td>( p_{1,0}(t) )</td>
</tr>
<tr>
<td>( R_2 )</td>
<td>( p_{2,0}(t) )</td>
</tr>
<tr>
<td>( \ldots )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( R_{M-1} )</td>
<td>( p_{M-1,0}(t) )</td>
</tr>
</tbody>
</table>

**Practical Implementation**
The signal generation and collection proceeds as follows: In a round-robin fashion, one active sensor at a time is activated as transmitter. The reflected signals are received at all the sensors: the activated sensor acts in pulse-echo mode, i.e. as both transmitter and
receiver, the other sensors act as passive sensors. Thus an \(M \times M\) matrix of signal
primitives is generated (Table 16.1).

The signal primitives are assembled into synthetic beamforming responses using the
synthetic beamformer algorithm equation (16.3). The delays, \(\delta_j\), are selected in such a
way as to steer the interrogation beam at a certain angle, \(\phi_0\). The synthetic-beam sensor
responses, \(w_i(t)\), synthesized for a transmitter beam with angle \(\phi_0\), are assembled by the
receiver beamformer into the total received signal, \(s_R(t)\), using the same delay as for the
transmitter beamformer. However, this method presumes that the target angle \(\phi_0\) is know
\textit{a priori}. In general applications, the target angle is not know, and needs to be determined.
Hence, we write the received signal as a function of the parameter \(\phi_0\), using the array unit
delay for the direction \(\phi_0\) as \(\Delta_\phi(\phi_0) = \frac{d}{c} \cos \phi_0\). In practical implementation of the method,
we have used a spline interpolation approach to implement the time shifts with accuracy
at times that fall in between the fixed values of the sampled time.

A coarse estimate of the target direction is obtained by using an azimuth sweep technique,
in which the beam angle, \(\phi_0\), is modified until the maximum received energy is obtained,
i.e.,

\[
\max E_R(\phi_0), \quad E_R(\phi_0) = \int_{t_p}^{t_p+b_0} |s_R(t, \phi_0)|^2 \, dt \tag{16.8}
\]

After a coarse estimate of the target direction is found \(\phi_0\), the actual round-trip time of
flight, \(\tau_{\text{TOF}}\), is calculated using an optimal estimator, e.g., the cross-correlation between
the received and the transmitted signal:
\[ y(\tau) = \int_{t_p}^{t_p + t_h} s_R(t) s_T(t - \tau) dt \]  

(16.9)

Then, the estimated $\tau_{\text{TOF}} = 2R/c$ is attained at the value of $\tau$ where $y(\tau)$ is maximum.

Hence, the estimated target distance is

\[ R_{\text{exp}} = c \tau_{\text{TOF}}/2 \]  

(16.10)

This algorithm works best for targets in the far field, for which the “parallel-rays” assumption holds.

For targets in the near and intermediate field, a more sophisticated self-focusing algorithm, that uses triangulation principles, is used. This algorithm is an outgrowth of passive-sensors target-localization methodologies (Silvia, 1987) The self-focusing algorithm modifies the delay times for each synthetic-beam response, $w_t(n)$, such that it maximizes the total response, by finding the focal point of individual responses, i.e., the common location of the defect that generated the echoes recorded at each sensor. For very close range targets, SAFT techniques (Shandiz and Gaydecki; 1999) are utilized.

The EUSR system consists of three major modules: the PWAS array, the DAQ module, and the Signal Processing module. A system diagram is shown in Figure 16.4.

A proof-of-concept system was built in Laboratory for Active Materials and Smart Structures (LAMSS) to evaluate the feasibility and capability of the EUSR system. Three specimens were used in the experiments. These specimens were 1220-mm (4-ft) square panel of 1-mm (0.040-in) thick 2024-T3 Al-clad aircraft grade sheet metal stock. One of the specimens was pristine, and was used to obtain baseline data. The other two were manufactured with simulated cracks. The size of the cracks were 19 mm ($\frac{3}{4}$-in) long,
0.127 mm (0.005-in) wide, and were placed: (a) broadside at \( R = 305 \) mm from the PWAS array; and (b) offside at \( R = 453 \) mm, \( \phi_0 = 130.3 \) deg. The EUSR system were used to detect the locations of the cracks.

![Diagram of EUSR LabVIEW program](image)

**Figure 16.4** Data flow diagram of the EUSR LabVIEW program

PWAS array is an array of small size, inexpensive, and non-intrusive piezoelectric-wafer active sensors. The transducers are made from American Piezo Ceramic Inc. APC850 piezo ceramic wafers, 7mm x 7mm in size. These transducers can work as both transmitters and receivers. The EUSR system tested in the LAMSS used an array consists of 9 PWAS placed on a straight line and spaced at distance \( l = \lambda/2 \), where \( \lambda \) is the wavelength of the guided wave propagating in the thin-wall structure. According to elementary wave mechanics, \( \lambda = c/f \), where \( c \) is the wave speed and \( f \) is the wave frequency. Our proof-of-concept experiments, performed on 1-mm gauge 2024-T3 aluminum alloy aircraft-grade sheet material, identified the first optimum excitation frequency at 300 kHz (S\(_0\) mode) with the corresponding wave speed \( c = 5.440 \) km/s, and
wavelength $\lambda = 18$ mm. Hence, the PWAS spacing for our proof-of-concept PWAS array was selected $d = 9$ mm (Figure 16.5).

![Diagram of PWAS array](image)

Figure 16.5 Proof-of-concept EUSR construction: (a) 4-ft square plate with 9-element PWAS array at its center; (b) details of the 9-element PWAS array with 7-mm square PWAS.

DAQ module uses a computer to control and collect data from a multi-channel pulser/receiver. The multi-channel pulser/receiver consists of (a) a HP33120A arbitrary signal generator, (b) a Tektronix TDS210 digital oscilloscope, and (c) a digital controlled signal switching unit. The HP33120A arbitrary signal generator was used to generate a 300 kHz Hanning windowed tone-burst excitation with a 10 Hz repetition rate. Under the Hanning windowed tone-burst excitation, one element in the PWAS array generates a package of elastic waves that spread out into the entire plate with a circular wave front (omnidirectional pattern.) The Tektronix TDS210 digital oscilloscope, synchronized with
the signal generator, was used to collect the response signals from the PWAS array. One of the oscilloscope channels was switched among the remaining elements in the PWAS array by using a digitally controlled switching unit. A LabVIEW computer program was developed to control the digital signal switching unit and record the data from the digital oscilloscope to a group of raw data files.

The signal-processing module reads the raw data files and process these data with EUSR algorithm. Although the EUSR algorithm is not computationally intensive, the large amount of data points in each signal made this step time consuming. Hence the resulting data is saved in a EUSR data file on computer for later retrieval. This data file also enables other programs to access the EUSR data.

![3-D surface]

Figure 16.6    Data visualization of the EUSR results.
After being processed, the data was transformed from time domain to 2-D plane domain. Based on the EUSR algorithm, the resulting data file is a collection of signals that represent the structure response at different angles, defined by the parameter $\phi$. In other words, they represent the response when EUSR scanning beam turned onto angle $\phi$.

Knowing the Lamb wave speed $c$, by using $r=ct$, this signal can be transformed from voltage $V$ vs. time $t$ to voltage $V$ vs. distance $r$. The signal detected at angle $\phi$ can be plotted on a 2-D plane at angle $\phi$. Since angle $\phi$ was stepped from 0 to 180°, at constant increments, the plots covered a half space. These plots generates a 3-D surface (Figure 16.6), which is a direct mapping of the structure being interrogated, with the $z$ value of the 3-D surface representing the response voltage from that $(x,y)$ location. Recall that the voltage is related to the reflection amplitude level, so this surface also represents the intensity of the reflections. If we present the $z$ value in color scale, then the 3-D surface is project to the 2-D plane, and the color of each point on the plane represents the intensity of the reflections.

The two pictures in Figure 16.7 show the EUSR results from the proof-of-concept system. The grids represent one half of the 4-ft square specimen plate. The shaded area is a projection of the 3-D surface; the amplitude is presented in color/grayscale intensity.

Figure 16.7 (a) is from the broadside specimen. The small area with darker color represents high amplitude echo (reflected wave) generated when the scanning angular beam intercepted the crack. From the picture, this area is located at a distance of 0.31 meters from the center of the plate and at 90 degrees. Recall that the simulated crack on this specimen is 305 mm (0.305 meters) from the PWAS array, and at $\phi_0 = 90$ degrees. The dark area in the EUSR result predicted the simulated crack perfectly.
Figure 16.7  Visualized EUSR results. The data are from specimens with crack at (a) broadside ($\phi_0 = 90$ deg) and (b) offside ($\phi_0 = 136$ deg).

Similarly, Figure 16.7 (b) is from the offside specimen, and the darker area predicted the simulated crack at a location $0.3\sqrt{2}$ meters from the center, 135 degrees, which estimated the real location of the simulated crack perfectly.
CHAPTER 17  CONCLUSIONS AND FUTURE WORK

The importance of structural health monitoring is more and more realized by many industries. The efforts of several generations of researchers have delivered many SHM methods. Commonly used methods often require bulky transducers, expensive instrumentations, and very time consuming when large areas of structures are to be interrogated. These requirements make the SHM very expensive and, in many cases, limit the application of SHM. The solution to this problem is to find a reliable SHM technology which is inexpensive and non-intrusive. The sensors can be permanently applied to vital structures to record the structural health signatures. When needed, data can be collected and analyzed with the aid of computer programs. When the possibility of defects is detected, thorough interrogation can be performed with high precision NDE methods. In the search of such technology, PWAS generated Lamb wave propagation method was found to be a good candidate for SHM. This dissertation addresses the major aspects of the development of this method:

(1) Theoretical development associated with the Lamb wave propagation method;

(2) Theoretical development associated with the PWAS transducers;

(3) Data processing methods and assessment; and demonstration of a prototype of this method. The three parts of the dissertation explained these three aspects in detail.
17.1 RESEARCH CONCLUSIONS
The theories of guided wave propagation were thoroughly studied. These theories were gradually developed in the last century by many researchers. They are very important for ultrasonics NDE. Guided waves are elastic waves that can propagate along some types of structure, i.e., wave guides. In NDE applications, one of the popular guided wave is Lamb wave. Lamb wave can propagate in thin wall structures, e.g., plates and pipes, which can be found on all kinds of structures. Some of the properties of Lamb wave made it favorable choice in thin wall structure NDE. A list of these properties is:

1. Lamb waves propagates long distance without losing too much amplitude;
2. Lamb waves have many modes with different wave structures (thickness distribution) which change with wave frequency and thickness of the wave guide;
3. Lamb wave modes can be selectively generated by tuning the size of the PWAS and frequency of excitation;
4. At extreme frequencies and/or wave guide thickness, Lamb wave demonstrate the properties of less complicated wave types, i.e., pressure, shear, and Rayleigh waves;
5. It can propagate omni-directionally with a circular wave front.

Starting with the basic elastic wave equations, the behavior of Lamb wave was investigated in the first part of this dissertation. Items 2 through 5 in the list were studied in-depth to provide a good understanding of Lamb wave.

Another important aspect of Lamb wave study is to explain how energy propagates with the Lamb wave. Furthermore, how does the amplitude distribute to each of the Lamb
wave modes? These information are also presented in part I of this dissertation. The importance of these aspects is that they give the theoretical foundation for determining the amplitude of Lamb wave under different loading conditions.

Lamb wave transducers are the direct application of Lamb waves theory. There are many types of Lamb wave transducers, but the majority of them are built based on one of the two mechanisms:

1. Using angled incident wave;

2. Applying force on the surface.

The first method often uses a wedge to make an incident pressure wave goes into the target in an angle. The Lamb wave mode can be selected by changing this angle. And the angle is calculated by using Snell’s law with the incident wave velocity and the desired Lamb wave mode velocity.

The second method directly uses the surface stress/strain distribution of Lamb wave modes, and generates a surface load similar to this distribution, hence selectively generates a desired Lamb wave mode. Comb transducer and interdigital transducer are examples of this method.

Conventional Lamb wave transducers are normally bulky and/or expensive due to the complicated construction of the transducer. A new type of transducer is PWAS transducer. PWAS are small unobtrusive piezoelectric transducers, which serve as active elements to generate Lamb waves by applying a strain on the surface of the wave guide (the second method). It can also be used as sensors to detect the presence of elastic waves.

A theoretical model of PWAS was provided in the second part of the dissertation.
In part II of the dissertation, the actuation and sensing equations of PWAS were derived. The factors that affect the operation of PWAS were discussed. Theoretical examples of PWAS under certain loading conditions were analyzed. The interaction between PWAS and thin plates were also derived.

Part III of the dissertation provides results of a series of experiments that validate using PWAS for Lamb wave generation and detection. The major findings in this part of the study are listed as following:

1. PWAS is capable of generating and receiving Lamb waves in thin plates and beams. The received signal is “loud and clear” at a long distance (> 4 ft.).

2. For low voltage (<20Vpp), low frequency (100kHz~1MHz) applications, no special designed equipments are needed. General purpose arbitrary function generator and digital oscilloscope were found sufficient. Considering the low cost of the PWAS itself, the system cost is greatly reduced compare with conventional ultrasonic systems.

3. It is important to choose a good excitation signal. Because the Lamb waves are dispersive, the received signal is distorted. A carefully designed excitation signal can concentrate the energy in a narrow bandwidth, hence eliminate the unwanted side band noise, and provide a cleaner signal.

4. The signal processing involves extracting the time of flight (TOF) information from the received signal. Because the good amplitude and clarity of the PWAS signals, this step is relatively simple. When needed, the envelope of the cross-
correlation of the excitation and received signal showed good performance in
extracting the TOF.

5. Experiments showed that PWAS are capable of generating omni-directional,
circular fronted Lamb waves in large plates.

6. The size of the PWAS and the frequency of excitation signal have significant
effects. By changing the size and/or frequency, optimum amplitude of specific
Lamb modes can be obtained.

7. With an array of PWAS, a 2-D scan of large plate can be achieved. Using the
phased array algorithm, the signals from \( N \) sensors can be assembled and plotted
in a 2-D plot. If the amplitude of the signal is represented by color, the presences
of reflectors are identified as distinctive colors on the plot. Hence the distance and
direction with respect to the center of the array can be determined. The PWAS
array also provides \( N^2 \)-time amplification to the received signal, which is very
good for signal analysis and reduces the system cost.

17.2 RECOMMENDED FUTURE WORK
The work presented in this dissertation reviewed the existing theories and methods that
are related to the PWAS and its interaction with host structures. Observations were made
on the complex structure of Lamb wave modes. The PWAS generated Lamb waves were
studied analytically and experimentally. But there are still questions to be solved to fully
discover the potential of PWAS for SHM. These questions fall into two major categories.
The first category is related to the nature of Lamb wave and other guided waves. And the
second category is related to the development of PWAS and PWAS array.
17.2.1 Future guided wave studies

Higher frequency can provide better resolution, but most of the researchers work in relatively low frequencies. The reason is that at high frequency, more Lamb wave modes will be generated. Due to the dispersive nature of the Lamb wave, more than a couple of the lowest Lamb wave modes (A₀ and S₀) will make the wave signal very complicated, hence more difficult to analyze. On the contrary, the complexity of the higher Lamb modes could have been used for the benefit of NDE. Besides the higher resolution, the complicated wave structure of higher modes can be used for detecting defects inside the material, especially for composite materials. Obviously, more study has to be done to discover these potentials.

Another issue is how to actuate Lamb waves with higher efficiency. As discussed in this dissertation and papers from other researchers, carefully chosen combination of PWAS size and frequency can tune the wave signal to larger amplitude. It was discovered that when the length of the PWAS is close to the half wavelength of Lamb wave, higher excitation efficiency is achieved.

In order to fully utilize the potential of Lamb wave and other guided waves, a more detailed study on the structure of guided waves will help to increase our knowledge. And in turn it will help to improve the development of guide wave NDE.

17.2.2 Future PWAS and PWAS array studies

PWAS was not only used in wave propagation NDE, but was also used in Electro-Mechanical (E/M) impedance health monitoring. The E/M impedance method is used to detect defects close to the PWAS. This is the area that can not be detected with pulse-echo method. A combination of these two methods can fully cover large areas of a
structure. More experiments need to be done to evaluate the capacity of such combination. And more application may be developed.

PWAS array was demonstrated in this dissertation, but only single target was simulated in the experiments. Multiple target, different orientation, and defects close to the boundary of structure (e.g., crack close to a rivet hole) were not covered. The 1-D array presented in the dissertation can only position targets in the 180-degree range (half plane). In order to realize full plane detection, 2-D array is required. Theoretical and experimental study need to be carried out to future develop the PWAS array design.
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