Chapter 1: Fundamentals of Vibrations

1 Basic Concepts in Vibrations

The study of vibrations, within the broader field of classical mechanics, is the investigation of oscillations that occur about an equilibrium point. Vibrations, both desired and undesired, are present in all mechanical systems and can be helpful (e.g. a soil sieve, rotary sander) or destructive (e.g. an aircraft frame in resonance). The oscillations that form a vibrating system may be periodic (e.g., pendulum) or random (e.g. turbulence in an airplane), or a combination of the two.

Vibrations impact our daily lives in a variety of ways, from the sound made by banjo strings that vibrates between 140 and 400 Hz to the 4-6 Hz vibration felt by a passenger in a car seat. The consideration of the vibrations and their associated mathematical modeling are an important factor in the design of mechanical systems. In the material that follows, the fundamental theories of vibration are presented and modeled using fundamental physical principles such as Newton’s three laws of motion. These models and analyzed using the mathematical tools of calculus and differential equations.

Review 1.1 Newton’s three laws of motion:

1. In an inertial frame of reference, an object either remains at rest or continues to move at a constant velocity, unless acted upon by a force.

2. In an inertial reference frame, the vector sum of the forces $F$ on an object is equal to the mass $m$ of that object multiplied by the acceleration of the object: $F = ma$. (It is assumed here that the mass $m$ is constant)

3. When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.
1.1 Single Degree-of-Freedom Systems

In its simplest form, the phenomenon of vibration is the exchange of energy between potential and kinetic energy. Therefore, a vibrating system must have a component that stores potential energy. This component must also be capable of releasing the energy as kinetic energy. This kinetic energy is stored in the movement of a mass where the measure of this movement is the velocity of the system and the continuous interchange between potential and kinetic energy is the vibration of the system. The simplest vibrating systems can be modeled as a single-degree-of-freedom (1-DOF) system. In a 1-DOF system, one variable can describe the motion of a system. Potential examples of 1-DOF systems include:

1. yo-yo
2. pogo stick
3. door swinging on axis
4. throttle (gas pedal)

Variables often used for describing 1-DOF systems are $x(t), y(t), z(t),$ and $\theta(t)$. Examples of 1-DOF systems are presented in figure 1 where the assumption of small displacements is made. Note: we will often drop the “$(t)$” for simplicity in this material.

![Figure 1: Examples of single degree of freedom (DOF) systems showing: (a) a vertical spring-mass system; (b) a simple pendulum; and (c) a rotational spring-mass system.](image-url)
1.1.1 Spring-Mass Model

“All models are wrong, but some are useful”
George E.P. Box (1919 - 2013)

Newtonian physics describes the motion of particles in terms of displacement \( x \), velocity \( \dot{x} \), and acceleration \( \ddot{x} \) vectors. Moreover, from Newton’s second law of motion says that the change in the velocity of mass in motion is a product of the force acting on the mass. A simple way to express this phenomenon is though a spring-mass model as presented in figure 2. These spring-mass models neglect the mass of the spring and concentrate all the mass of the system into a single point. Note that in this case the force vector and mass-acceleration vectors lie on the same axis and as such are collinear. Therefore, these vectors can be easily treated as scalers simplifying the math used in the modeling of the system.

Figure 2: A single-degree-of-freedom (1-DOF) spring mass model showing: (a) annotated schematic of a mass-spring system; and (b) the equivalent free-body diagram represented as a point-mass system.

1.1.2 Linear Springs

Springs are mechanical devices that store energy, moreover, ideal spring is a theoretical representation of this mechanical device that is massless and responds with a linear increase in force for a unit increase in displacement (i.e. \( F = kx \)). For simplicity, the spring in the spring-mass model considered here is assumed always ideal linear springs. A graphical representation of the idealized linear spring is presented in figure 3 where a unit force \( F \) applied to the free end of the spring results in a unite displacement \( x \) of the spring. The resulting mathematical relationships, \( F = kx \), is known as Hooke’s Law. Nonlinear springs add considerable complexity
to the modeling of spring-mass systems, therefore, these are not considered in this introductory work.

1.2 Equivalent Stiffness

The generalized concept of stiffness can be directly related to mechanical systems and structural components through Hooke’s law.

**Review 1.2** Hooke’s Law states that the force \( F \) needed to extend or compress a spring by some distance \( x \) scales linearly with respect to that distance. This law can be extend to tensional stress of an uniform and elastic bar where the length, area, and Young’s modulus of the bar are represented by \( l, A, \) and \( E, \) respectively. Knowing the tensile stress in the bar:

\[
\sigma = \frac{F}{A} \tag{1}
\]

and the definition of strain:

\[
\varepsilon = \frac{\Delta l}{l} \tag{2}
\]
Hooke’s law can be expanded to represent an uniform and elastic bar:

\[ \sigma = E\varepsilon \]  

(3)

It follows that the change in length \(\Delta l\) can be expressed as:

\[ \Delta l = \varepsilon l = \frac{FL}{AE} \]  

(4)

**Note:** Hooke’s law is often expressed using the convention that \(F\) is the restoring force exerted by the spring on the applied force at the free end. Using this expression, the equation for Hooke’s Law becomes:

\[ F = -kx \]  

(5)
since the direction of the restoring force is opposite the spring displacement.

1.2.1 Equivalent Stiffness of Structural Systems

For a rod with a uniform cross-section, a direct representation of the system can be developed as expressed in figure 4 where the vibration along the axis of the rod is to be considered. The stiffness of the rod, \(k\), is a measure of the resistance offered by an elastic body to deformation.

![Figure 4: Equivalency between a vertical bar with mass attached to the bottom and a spring-mass model of the system](image)

For this 1-DOF system, the equation of a spring can be rearranged such that the stiffness can be defined as:

\[ k = \frac{F}{x} \]  

(6)
The stiffness of the spring can be more closely related to materials properties of the bar $A$, $E$, and $l$ considering that Hooke’s Law for the uniform tension on a bar can be expressed as:

\[ \sigma = E \varepsilon \quad (7) \]

This expression can be expanded into the form:

\[ \frac{F}{A} = E \left( \frac{x}{l} \right) \quad (8) \]

rearranging the terms and recalling the expression $k = \frac{F}{x}$ leads to:

\[ k = \frac{EA}{l} \quad (9) \]

In a similar fashion, we can also solve the equivalent system for a mass at the end of a cantilever beam.

From engineering mechanics we can compute the deflection at the point of a beam $\delta$ with a point load $P$. This expression is typically expressed as:

\[ \delta = \frac{Pl^3}{3EI} \quad (10) \]

If we transform this matrix into our variable system by exchanging $P$ for $F$ and $\delta$ for $x$. Thereafter, the point load is replaced with the equivalent force $F$ generated by the mass and the pull of gravity($mg$). As before, knowing that the stiffness of the system can be expressed as $x = F/x$ we can show that:

\[ k = \frac{3EI}{l^3} \quad (11) \]
Example 1.1 Considering the rod diagrammed below; calculate an equivalent spring constant for the rod using the length of the rod $l$, its area $A$, and Young’s modulus $E$ for a compressive force $F$ that compresses the rod a distance $x$. Additionally, is a linear spring a useful model for a rod under compression? What if the rod is under tension?

![Compressed cantilever rod](image)

**Solution:** The rod shortens by a distance $x$ under the axial force $F$, this can be related to the equation of a linear spring $F = kx$ by recalling from solid mechanics that the elongation (or shortening) of a rod is expressed as

$$x = \frac{x}{l} = \frac{\varepsilon l}{E} = \frac{\sigma l}{E} = \frac{F l}{AE}$$  \hspace{1cm} (12)

where $\varepsilon = \frac{x}{l}$ is the strain value and $\sigma = \frac{F}{A}$ is the stress induced in the rod. Combining this expression with the equation of a linear spring yields:

$$k = \frac{F}{x} = \frac{AE}{l}$$  \hspace{1cm} (13)

As per the usefulness of the linear spring to represent an axial rod under compression or tension, this would be application-specific but could generally be considered an excellent first-order approximation.

1.2.2 Springs in Series and Parallel

In many cases, it becomes necessary to model a mechanical system as a set of springs (e.g., a composite material, a table with multiple legs). For these systems, or for systems with more than one spring acting on a body, equivalent stiffness can be calculated as:
Figure 6: Equations for calculating the equivalent stiffness of two springs ($k_1$ and $k_2$); (a) in series; and (b) in parallel.

These are derived considering the displacement $\delta$ of the systems. For two springs in series:

\[
\delta_{ac} = \delta_{ab} + \delta_{bc} \quad (14)
\]

where the total displacement is

Using the equation for stiffness $k = F/\delta$, this converts to:

\[
\frac{F}{k_{ac}} = \frac{F}{k_1} + \frac{F}{k_2} \quad (15)
\]

As $F$ is the same throughout the system, we can cancel out $F$. Solving for the equivalent stiffness yields:

\[
k_{ac} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}} \quad (16)
\]

Similarly for a system of springs in parallel:

Figure 7: Two springs $k_1$ and $k_2$ combined in series.
The displacement in both springs is the same, so the total displacement is
\[ \delta_{ab} = \delta_1 = \delta_2 = \delta \]  
(17)

The forces in the direction of spring elongation sum to zero, therefore:
\[ F_{ab} = F_1 + F_2 \]  
(18)

Substituting the displacement and stiffness into the force equation yields:
\[ \delta k_{ab} = \delta k_1 + \delta k_2 \]  
(19)

distrs this simplifies to:
\[ k_{ab} = k_1 + k_2 \]  
(20)

**Example 1.2** Calculate the equivalent stiffness of the following system:

The springs are combined as shown, using the equations defined before. Now, considering that the displacement (\( \delta \)) of the top spring, and the bottom spring are the same we can state the total stiffness \( k \), is the summation of the two. Therefore,
where the final addition, \((k_1 + k_2) + (k_5 + \frac{1}{k_3 + k_4})\) is applied at two springs in parallel as each spring is connected between the mass and the fixity. Rearranging this new expression to get a common denominator:

\[
k = \frac{(k_1 + k_2 + k_5)(k_3 + k_4) + k_3k_4}{k_3 + k_4}
\]  

(21)

1.3 Equation of Motion for an Oscillating System

An Equation of Motion (EOM) is an equation that provides a basis for modeling a vibrating system about its equilibrium point and relates the transfer of the potential energy from the spring to the kinetic energy mass. In developing the EOM we assume that any surfaces are frictionless and as such, no energy is extracted from the vibrating system. Referencing the 1-DOF system in figure 9(a), and assuming the mass only moves in the x direction, the only force acting on the mass in the x direction is the force that results from the elongation of the spring as annotated in figure 9(b). Therefore, the sum of forces in along the x axis must equal the mass \((m)\) times the acceleration of the mass \((a \dot{x})\).

![figure 9](image-url)  

Figure 9: A spring mass model of a 1-DOF system showing: (a) a schematic of the system; (b) free-body diagram of the system at its initial position.
Considering that positive displacements are to the right, the standard form of the equation of motion for an undamped system without any excitation is expressed as:

\[ s_1 \ddot{x} + s_2 x = 0 \]  \hspace{1cm} (22)

where \( s_1 \) and \( s_2 \) are constants to be determined for the specific system. A systematic approach to obtaining free-body diagram (FBD) of a system under vibration can be expressed in three steps:

1. Draw a free-body diagram (FBD) at the system’s equilibrium and displaced position (without a displacing force).
2. Apply Newton’s second law to both FBDs (equilibrium and displaced).
3. Combine the equations to write the EOM in standard form with the forcing component on the right-hand side. For free vibration, the forcing component is 0.

Solving these three steps for 1-DOF system presented in figure 9 results in the EOM:

\[ m \ddot{x} + kx = 0 \]  \hspace{1cm} (23)

**Review 1.3** A second-order linear homogeneous differential equation has the form:

\[ a \ddot{x} + b \dot{x} + cx = 0 \]  \hspace{1cm} (24)

The EOM for a 1-DOF system under a free vibration is a a second-order differential equation due to acceleration (\( \ddot{x} \)) being the second derivative of displacement (\( x \)) and homogeneous as the forcing function (right-hand side of the equations) is zero. In EOM’s current form, \( m = k \), \( b = 0 \), and \( c = k \). In future work, \( b \) will account for damping in the vibrating system.

**Example 1.3** Considering the system:
Step-1 Define the direction of displacement, and draw the FBD for the equilibrium and displaced state.

The equation for the equilibrium state is:

$$\sum F_x = 0 \quad (25)$$

and in the displaced state:

$$\sum F_x = -kx \quad (26)$$

This equation does not equal zero as the FBD does not account for the restoring force.

Step-2 Apply Newton’s second law (we want to store energy in the kinetic state) of motion to the sum of forces for the displaced position we get:

$$ma = m\ddot{x} = \pm \sum F_x = -kx \quad (27)$$

$$m\ddot{x} = -kx \quad (28)$$

Step-3 Rearrange in the Equation to construct an EOM:

$$m\ddot{x} + kx = 0 \quad (29)$$

Example 1.4 Some systems will have an initial displacement, as the system will oscillate around this position we need to define the EOM about this position. Considering the system:
Step-1 Define the direction of displacement, and draw the FBD for the equilibrium and displaced state.

The equation for the equilibrium state is:

\[ +\downarrow \sum F_x = mg - k\delta = 0 \]  
(30)

and in the displaced state:

\[ +\downarrow \sum F_x = mg - k(\delta + x) \]  
(31)

This equation does not equal zero as the FBD does not account for the restoring force.

Step-2 Apply Newton’s second law (we want to store energy in the kinetic state) of motion to the sum of forces for the displaced position we get:

\[ m\ddot{x} = +\downarrow \sum F_x = mg - k\delta - kx \]  
(32)

We can than use the information from the equilibrium state to cancel out some terms, this becomes:

\[ m\ddot{x} = -kx \]  
(33)
Step-3 Rearrange in the Equation to construct an EOM:

\[ m\ddot{x} + kx = 0 \]  

(34)
2 Free Vibration of Single-Degree-of-Freedom Systems

Vibrations (i.e. the exchange of potential and kinetic energy) requires oscillatory motion that may repeat itself regularly or irregularly. A motion that is repeated on time intervals is called periodic motion. If this motion has a single frequency and amplitude it is called simple harmonic motion and represents the most basic form of oscillatory motion as depicted in figure 1. For a 1-DOF system simple harmonic motion is defined as a periodic motion where the restoring force is directly proportional to the displacement and acts in the direction opposite to that of displacement.

![Oscillating Mass](image)

Figure 1: Oscillatory motion for a single degree of freedom system showing (a) periodic motion; and (b) simple harmonic motion.

Given the nature of simple harmonic motion, constant amplitude and frequency, the wave starting at the origin \(O\) can be modeled at a point on the end of a vector with length \(A\) rotating at a constant angular velocity \(\omega_n\) where the angle from the origin of the vector is \(\phi\), defined as \(\phi = \omega t\). Where \(\omega\) is the lowercase Greek letter Omega and \(\phi\) is the lowercase Coptic letter phi. This is similar to a Greek phi (\(\varphi\)) and either can be used in this context. The subscript \(n\) on \(\omega\) denotes that this frequency relates to the natural frequency of the system, the only frequency in simple harmonic motion. A visualization of the harmonic motion obtained from projecting the point on the edge of a vector onto the \(\omega_n t\) space is presented in figure 2.
2.1 Mathematical Modeling of Free Vibration

The development of a mathematical model for a system under free vibration would enable the practitioner to predict, or model, the vibrating system of interest. Therefore, considering that the following system,

\[ m\ddot{x}(t) + kx(t) = 0 \] (1)

can be modeled expressed with the following EOM

\[ m\ddot{x}(t) + kx(t) = 0 \]

it becomes prudent to solve this homogeneous ordinary differential equation (ODE) to obtain a model of the vibrating system. The simplest method for solving an ODE is to propose a solution based on observations of a vibrating physical system. Figure 4 reports and annotates the key components from an observation of a vibrating system.
where $x_0$ and $v_0$ are the is the displacement and velocity at $t=0$ (i.e. the initial displacement).

A mathematical expression can now be formulate to represent the observed simple harmonic motion. This expression can be based on the projection of a point on a vector (transposed into the time domain) or assembled from constituent parts as done in what follows. Solving for a location $x$, at a time $t$; $x(t)$, the various characteristics of the expression can be identified:

2.1.1 Solve for the Natural Frequency ($\omega_n$) of the System

- System oscillates $\rightarrow$ a sin function models this
- System oscillates at different speed $\rightarrow$ use a parameter to adjust $\omega_n$ in rad/s.
- Systems have different amplitudes $\rightarrow$ use a parameter to adjust $A$ in meters.
- System has different starting points $\rightarrow$ use a parameter to adjust $\phi$ in rad.

Using these four constituent components, an equation can be proposed:

$$x(t) = A \sin(\omega_n t + \phi)$$  \hspace{1cm} (2)

Take the derivative to get velocity:

$$\dot{x}(t) = A \omega_n \cos(\omega_n t + \phi)$$  \hspace{1cm} (3)

Take the derivative again to get acceleration:

$$\ddot{x}(t) = -A \omega_n^2 \sin(\omega_n t + \phi)$$  \hspace{1cm} (4)
Substituting $x$ and $\ddot{x}$ into the EOM for the considered 1-DOF system ($m\ddot{x}(t) + kx(t) = 0$) yields:

$$m( - A\omega^2_n \sin(\omega_n t + \phi)) + k(A\sin(\omega_n t + \phi)) = 0 \quad (5)$$

Thereafter, dividing both sides by $A\sin(\omega_n t + \phi)$ results in the expression:

$$-m\omega^2_n + k = 0 \quad (6)$$

This expression can be rearranged into the more useful standard form:

$$\omega_n = \sqrt{\frac{k}{m}} \quad (7)$$

Equation 7 represents a solution to the EOM presented in equation 1. This solution is not in the form of an ODE so, therefore, we can experientially prove that this is the correct solution. For example, we could build a systems with known mass and stiffness and measure the natural frequency of the system. Equation 7 equation leads to:

$$T = \frac{2\pi}{\omega_n} \quad (8)$$

where $T$ is the period of oscillations and

$$f_n = \frac{\omega_n}{2\pi} \quad (9)$$

where $f_n$ is the frequency of the oscillations.

### 2.1.2 Solve for Initial Phase ($\phi$) of the System

The EOM is a second-order ODE so there needs to exist two initial conditions (constants) to solve it. For the systems under considerations, the displacement ($x$) and velocity ($\dot{x}$ or $v$) at $t = 0$ are the initial conditions. For simplicity, these are written as

$$x(0) = x_0 \quad (10)$$

$$\dot{x}(0) = v(0) = v_0 \quad (11)$$

Setting the equation to its initial state $t = 0$, the equations for displacement and velocity can be simplified to:

$$x(0) = x_0 = A\sin(\omega_n 0 + \phi) = A\sin(\phi) \quad (12)$$
\[ \dot{x}(0) = v_0 = A\omega_n \cos(\omega_0 + \phi) = A\omega_n \cos(\phi) \] (13)

Thereafter, mathematical meanings for \( \phi \) and \( A \) can be derived. To do this, \( \phi \) can be solved for by rearranging equations 12 and 13 for \( A \):

\[
A = \frac{x_0}{\sin(\phi)}
\] (14)

and:

\[
A = \frac{v_0}{\omega_n \cos(\phi)}
\] (15)

Setting these two equations equal to each other cancels out \( A \) and creates:

\[
\frac{x_0 \omega_n}{\sin(\phi)} = \frac{v_0}{\cos(\phi)}
\] (16)

therefore:

\[
\frac{x_0 \omega_n}{v_0} = \frac{\sin(\phi)}{\cos(\phi)}
\] (17)

finally:

\[
\phi = \tan^{-1}\left(\frac{x_0 \omega_n}{v_0}\right)
\] (18)

2.1.3 Solve for Amplitude \((A)\) of the System

The amplitude of the vibrating system \((A)\) is solved for in a similar manner to \( \phi \) where the expressions for \( x \) and \( \dot{x} \) are solved for at \( t = 0 \) and rearranged as to isolate \( \phi \). This operations results in the equations:

\[
\sin(\phi) = \frac{x_0}{A}
\] (19)

and:

\[
\cos(\phi) = \frac{v_0}{\omega_n A}
\] (20)

From these equations a value for \( \phi \) can be obtained knowing that \( \sin(\phi)^2 + \cos(\phi)^2 = 1 \). Therefore:

\[
\left(\frac{x_0}{A}\right)^2 + \left(\frac{v_0}{\omega_n A}\right)^2 = 1
\] (21)

multiplying each expression by 1 (also expressed as \( \frac{\omega_n}{\omega_n} \)), gives the equation:

\[
\left(\frac{\omega_n}{\omega_n}\right)^2 \left(\frac{x_0}{A}\right)^2 + 1 \left(\frac{v_0}{\omega_n A}\right)^2 = 1 \times 1
\] (22)
which becomes:

\[
\left( \frac{\omega_n x_0}{\omega_n A} \right)^2 + \left( \frac{v_0}{\omega_n A} \right)^2 = 1 \quad (23)
\]

Further simplification is obtained by multiplying each side by \((\omega_n A)^2\) to obtain:

\[
\omega_n^2 x_0^2 + v_0^2 = A^2 \omega_n^2 \quad (24)
\]

Solving for \(A\), this expression rearranges to:

\[
A = \sqrt{x_0^2 + \left( \frac{v_0}{\omega_n} \right)^2} \quad (25)
\]

### 2.1.4 Response for Simple Harmonic Motion

The time-varying displacement of a 1-DOF vibrating system under free response is expressed by the equation \(x(t) = A \sin(\omega_n t + \phi)\). Substituting in the expressions for \(A\) and \(\phi\) results in:

\[
x(t) = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} \sin \left( \omega_n t + \left( \tan^{-1} \left( \frac{x_0 \omega_n}{v_0} \right) \right) \right) \quad (26)
\]

This equation provides a mathematical solutions that relates displacement of the mass to the initial conditions \(x_0\) and \(v_0\). The solution is considered a free response because no input is applied after \(t=0\). The relationship between the initial conditions \((x_0\) and \(v_0\)) and the amplitude and phase of the response can be expressed using the Pythagorean theorem, \(a^2 + b^2 = c^2\), as annotated in figure 5.

![Figure 5: Trigonometric relationship between the initial conditions \((x_0\) and \(v_0\)), amplitude \(A\), and phase \(\phi\) for free vibration of a 1-DOF system.](image-url)
2.1.5 Special Considerations for No Initial Velocity ($v_0 = 0$)

Upon close inspection of the temporal solution in equation 26, it becomes evident that any system without initial velocity (i.e. $v_0 = 0$) results in an undefined number for $(x_0 \omega_n)/v_0$. A solution to this challenge lies in the fact that limit of $\tan^{-1}(x)$ approaches $-\pi/2$ at $-\infty$ and $\pi/2$ at $\infty$, as depicted in figure 6. Therefore, the solution at $-\infty$ and $\infty$ is undefined, resulting in the expression:

$$\left( \frac{x_0 \omega_n}{v_0} \right) = \pm \frac{\pi}{2}, \text{ when } v_0 = 0 \quad (27)$$

This step is applied in IEEE floating point arithmetic (IEEE 754) and results in either $\pi/2$ or $\pm \pi/2$ depending on the rounding format used. From the practitioner’s side, it becomes important to recognize the situation $v_0 = 0$ and correct this value as needed.

![Figure 6: Response of $\tan^{-1}$ (or arctan) for $x$=-60 to 60 showing that the $\tan^{-1}$ is undefined as $x$ approaches $-\infty$ and $\infty$.](image)

**Example 2.1** A vehicle wheel, tire, and suspension can be modeled as a SDOF spring and mass as depicted below: The mass of the wheel and tire is measured to be 300 kg and its frequency of oscillation is observed to be 10 rad/sec. What is the stiffness of the wheel assembly?
Figure 7: Modeling of a vehicle wheel, tire, and suspension showing: (a) Graphical representation; and (b) a spring-mass model.

Solution: Considering:

\[ \omega_n = \sqrt{\frac{k}{m}} \]  \hspace{2cm} (28)

therefore, \( k = m\omega_n^2 = (300 \text{ kg})(10 \text{ rad/s})^2 = 30 \text{ KN/m} \). Note: radians are a dimensionless quantity and as such the units of \( m\omega_n^2 \) become \( \frac{\text{kg}}{\text{s}^2} \cdot \frac{\text{m}}{\text{m}} \) where the unit value \( \frac{\text{m}}{\text{m}} \) is added such that the stiffness of the spring can be expressed as \( \frac{\text{kg} \cdot \text{m}}{\text{s}^2} \cdot \frac{1}{\text{m}} = \frac{\text{N}}{\text{m}} \).

Example 2.2 Consider the following 1-DOF system, where \( k = 857.8 \text{ N/m} \) and \( m = 49.2 \times 10^{-3} \text{ kg} \), calculate the natural frequency in rad/s and Hz. Also find the period of oscillations and the maximum displacement if the spring is initially displaced 10 mm with no initial velocity.

Figure 8: 1-DOF spring-mass system.
Solution:

\[
\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{857.8}{49.2 \times 10^{-3}}} = 132 \text{ rad/sec} \quad (29)
\]

In Hz, this is:

\[
f_n = \frac{\omega_n}{2\pi} = 21 \text{ Hz} \quad (30)
\]

The period is:

\[
T = \frac{2\pi}{\omega_n} = 0.0476 \text{ s} \quad (31)
\]

The maximum displacement will happen when \(\sin(\omega_nt + \phi) = 0\), therefore, the value of \(A\) is the maximum displacement. For an undamped system, \(A = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n}\),

\[
A = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} = \frac{\sqrt{132^20.01^2 + 0^2}}{132} = 0.01 \text{ m} \quad (32)
\]

2.2 General Solution for Vibrating Systems

The EOM for a vibrating system has many solutions and can be expressed in various forms including a general solution. These forms offer different mathematical approaches to solve the same 1-DOF spring-mass system and relate to each other through Euler’s equations.

Review 2.1 Vibration analysis uses complex numbers to solve the EOM’s differential equation. In this text the imaginary number is termed \(j\) (sometimes referred to as \(i\)): such that:

\[
j = \sqrt{-1} \quad (33)
\]

and:

\[
j^2 = -1 \quad (34)
\]

A general complex number, \(x\), can be expressed as:

\[
x = a + bj \quad (35)
\]

here, \(a\) is referred to as the real number and \(b\) is the imaginary part of the number \(x\). Such complex numbers can be represented in the complex plane,
also called a Argand plot. The absolute value or modules is defined as $|x|$ presented on the complex plot.

Figure 9: A conjugate pair of numbers ($A$ and $A^*$) represented on the complex plane.

$A$ and $A^*$ prime are complex conjugate pairs. In mathematics, the complex conjugate of a complex number is the number with an equal real part and an imaginary part equal in magnitude but opposite in sign. In other words, a conjugate pair is $a + bj$ and $a - bj$.

**Definition** - con·ju·gate (adjective): Coupled, connected, or related.

**Review 2.2** Euler’s (pronounced oy-ler) formula, named after Swiss engineer and mathematician Leonhard Euler (1707-1783), is a mathematical formula in complex analysis that establishes the fundamental relationship between the trigonometric functions and the complex exponential function. Euler’s formula states that for any real number $x$,

$$e^{j\psi} = \cos(\psi) + j\sin(\psi)$$  \hfill (36)

where $j = \sqrt{-1}$. This equation can also be expressed as:

$$e^{-j\psi} = \cos(\psi) - j\sin(\psi)$$  \hfill (37)
This can be expressed in terms of polar coordinates as:

Figure 10: Euler’s formula illustrated on the unit circle in the complex plane.

2.2.1 Formulating the General Solution for a 1-DOF Spring-Mass System

We can also solve the following EOM as an elementary differential equation:

\[ m\ddot{x} + kx = 0 \quad (38) \]

in a more analytical manner using the theory of elementary differential equations. To do this is the form:

\[ x(t) = ae^{\lambda t} \quad (39) \]

is assumed where, \( a \) and \( t \) are nonzero constants that need to be determined. Using successive differentiation, the assumed solution becomes:

\[ \dot{x}(t) = \lambda ae^{\lambda t} \quad (40) \]

and

\[ \ddot{x}(t) = \lambda^2 ae^{\lambda t} \quad (41) \]

therefore, \( m\ddot{x}(t) + kx(t) = 0 \) becomes:

\[ m\lambda^2 ae^{\lambda t} + kae^{\lambda t} = 0 \quad (42) \]

Next, the above expressions is divide by \( ae^{\lambda t} \) to obtain the characteristic equation:

\[ m\lambda^2 + k = 0 \quad (43) \]
This can be done because \( ae^{\lambda t} \) is never zero, therefore, the expressions is never divide by zero. The quadratic formula gives us:

\[
\lambda = \pm \sqrt{-\frac{k}{m}} = \pm \sqrt{\frac{k}{m}} j = \pm \omega_n j
\]  

(44)

remember that \( \omega_n = \sqrt{\frac{k}{m}} \). Notice that the \( \pm \) tells us there are two solutions to this problem. So, putting \( \lambda \) back into the assumed solution results in two solutions (one positive, one negative):

\[
x(t) = a_1 e^{+\omega_n j t}
\]

(45)

and

\[
x(t) = a_2 e^{-\omega_n j t}
\]

(46)

As these solutions only consider, and are only valid for, linear systems, the sum of the solutions is also a solution. This simplification results in:

\[
x(t) = a_1 e^{+\omega_n j t} + a_2 e^{-\omega_n j t}
\]

(47)

where \( a_1 \) and \( a_2 \) are complex valued constants of integration.

**Example 2.3** Show that \( x(t) = a_1 e^{+\omega_n j t} + a_2 e^{-\omega_n j t} \) is equal to \( A \sin (\omega_n + \phi) \).

**Solution:** This equation was derived using Euler’s formula and it can be shown that this equation is equivalent to the \( A \sin (\omega_n + \phi) \). To recover the previously assumed solution, the knowledge that \( a_1 \) and \( a_2 \) are complex congregate pairs and as such the magnitude can be expressed as \( a_1 = a_2 \) is leveraged. Using Euler’s polar notation, \( a_1 \) and \( a_2 \) can be expressed as

\[
a_1 = a_2 = ae^{j\psi}
\]

(48)

where \( a \) and \( \psi \) are real numbers, the equation becomes:

\[
x(t) = ae^{j(\omega_n t + \psi)} + ae^{-j(\omega_n t + \psi)}
\]

(49)

this becomes:

\[
x(t) = a(e^{j(\omega_n t + \psi)} + e^{-j(\omega_n t + \psi)})
\]

(50)
Remembering Euler’s equations from before, this becomes:

\[ x(t) = a \left( \cos(\omega_n t + \psi) + j \sin(\omega_n t + \psi) + \cos(\omega_n t + \psi) - j \sin(\omega_n t + \psi) \right) \]  

(51)

combining the “cos” terms and canceling out the “sin” terms this becomes:

\[ x(t) = 2a \cdot \cos(\omega_n t + \psi) \]  

(52)

This is equivalent to \( x(t) = A \sin(\omega_n t + \phi) \) considering that \( A = 2a \) and knowing \( \sin(\phi) = \cos(\phi + \psi) \). To expand, this is because the sin and cos are only differentiated by a phase shift.

Next, a general solution for the EOM is obtained. Using the previous solution:

\[ x(t) = a_1 e^{+\omega_njt} + a_2 e^{-\omega_njt} \]  

(53)

we can expand this into the form:

\[ x(t) = a_1 (\cos(\omega_n t) + j \sin(\omega_n t)) + a_2 (\cos(\omega_n t) - j \sin(\omega_n t)) \]  

(54)

using trigonometric functions. This equates to:

\[ x(t) = (a_1 + a_2) \cdot \cos(\omega_n t) + (a_1 - a_2) j \cdot \sin(\omega_n t) \]  

(55)

As \( x(t) \) is always real, \( A_1 \) and \( A_2 \) can be defined as:

\[ A_1 = (a_1 + a_2) \]  

(56)

and

\[ A_2 = (a_1 - a_2) j \]  

(57)

Lastly, as the general solution is written as:

\[ x(t) = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) \]  

(58)

This is the general solution for the EOM \( (m\ddot{x} + kx = 0) \) of the considered oscillating system where \( A_1 \) and \( A_2 \) are defined as:

\[ A = \sqrt{A_1^2 + A_2^2} \]  

(59)

and

\[ \phi = \tan^{-1} \left( \frac{A_1}{A_2} \right) \]  

(60)
These are obtained from a trigonometric relationship, similar to that used before:

\[ A = \sqrt{\frac{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}{\omega_n}} = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2} \quad (61) \]

\[ \phi = \tan^{-1}\left(\frac{x_0\omega_n}{v_0}\right) \quad (62) \]

**Example 2.4** Using the general solution:

\[ x(t) = A_1\cos(\omega_n t) + A_2\sin(\omega_n t) \quad (63) \]

Calculate the values of \(A_1\) and \(A_2\) in terms of their initial conditions \(x_0\) and \(v_0\).

**Solution:** Knowing the following for \(x\) and \(\dot{x}\):

\[ x(t) = A_1\cos(\omega_n t) + A_2\sin(\omega_n t) \quad (64) \]

\[ \dot{x}(t) = -A_1\omega_n\sin(\omega_n t) + A_2\omega_n\cos(\omega_n t) \quad (65) \]

Now apply the initial conditions, \(x(0) = 0\) and \(v(0) = 0\), this yields:

\[ x(0) = x_0 = A_1 \quad (66) \]

\[ \dot{x}(0) = v_0 = A_2\omega_n \quad (67) \]
Solving for $A_1$ and $A_2$ shows us:

\[ A_1 = x_0, \text{ and } A_2 = \frac{v_0}{\omega_n} \quad (68) \]

thus:

\[ x(t) = x_0 \cos(\omega_n t) + \frac{v_0}{\omega_n} \sin(\omega_n t) \quad (69) \]

### 2.2.2 Solution of 1-DOF System in Three Forms

Form one, for $m\ddot{x} + kx = 0$ subject to nonzero initial conditions can be written as:

\[ x(t) = a_1 e^{+\omega_n j t} + a_2 e^{-\omega_n j t} \quad (70) \]

where $a_1$ and $a_2$ are complex terms. Form two is:

\[ x(t) = A \sin(\omega_n t + \phi) \quad (71) \]

while form three is:

\[ x(t) = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) \quad (72) \]

where $A$, $\phi$, $A_1$, and $A_2$, are all real-valued constants. Each set of constants can be related to each other by:

\[ A = \sqrt{A_1^2 + A_2^2} \quad \phi = \tan^{-1}\left(\frac{A_1}{A_2}\right) \quad (73) \]

\[ A_1 = (a_1 + a_2) \quad A_2 = (a_1 - a_2) j \quad (74) \]

\[ a_1 = \frac{A_1 - A_2 j}{2} \quad a_2 = \frac{A_1 + A_2 j}{2} \quad (75) \]

Which follow from trigonometric identities and the Euler’s formulas.

### 2.3 Damping

The response of a spring-mass system predicts that a system will oscillate indefinitely. However, we know that this is not true from observing real-world solutions. So based on real-world observations and mathematical conveniences, we need to add a term that will remove “energy” from the system with time. To do this the idea of the ideal dashpot is introduced. A linear dashpot is diagrammed in figure 12 and is a mechanical device that resists motion via viscous friction and therefore
converts the mechanical energy of the system into thermal energy that is dissipated.

\[ f_c = cx \]  \hspace{1cm} \text{(76)}

the constant \( c \), called the damping coefficient, has the units of kg/s. Dashpots are a mathematical representation of viscous dampers installed in automobiles, aircraft, structures, and other mechanical devices. However, all systems have inherent damping not just systems with physical dampers. The spring-mass system can be used as a representation of real-world systems with inherent damping as demonstrated by the rubber engine mount depicted in figure 13.

Depending on the amount of damping present in a system, the temporal response of the system will represent itself in various ways, as represented in figure 14. To reiterate, an undamped case will oscillates around the equilibrium and does not decay. If a limited amount of damping is present in a system it will oscillates...
around the equilibrium and slowly decay with time to the equilibrium position, this is termed underdamped. If an excessive amount of damping is present, the system will not oscillate but decay directly to the equilibrium position, this is termed the overdamped case. Lastly, there exists a special case that results in the system converging as quickly as possible to the equilibrium position without oscillations; this case is termed the critically damped case. Furthermore, the amount of damping required to obtain a critically damped system is the damping value that the underdamped and overdamped cases for a specific system. To recap, the key types of damping are:

- **Undamped** - Oscillates around the equilibrium and does not decay.
- **Underdamped** - Oscillates around the equilibrium and slowly decays and is the most common case.
- **Overdamped** - Does not pass the equilibrium position and is a simple a decay with no oscillation.
- **Critically damped** - Provides the quickest approach to zero amplitude for a damped oscillator.

![Temporal responses for the three types of damping](image)

Figure 14: Temporal responses for the three types of damping: underdamped, over damped, and critically damped.
2.3.1 Modeling Vibrating Systems with Damping

The spring-mass system of chapter 1 can be expanded to a spring-dashpot-mass system that considers the damping component of the system. A mathematical model of the spring-dashpot-mass system can be developed for the case present in figure 15. Using the FBD for the system, it can conclude that the EOM for this system:

\[ m \ddot{x}(t) = -f_c - f_k \]  \hspace{1cm} (77)

Rearranging into standard form and concerting forces into parameters \( c \) and \( k \) results in:

\[ m \ddot{x}(t) + c\dot{x}(t) + kx(t) = 0 \]  \hspace{1cm} (78)

This system is subject to the same initial conditions as before, \( x(0) = x_0 \) and \( \dot{x}(0) = v_0 \). Again, choosing to model it this way for convinces, so let’s solve it in a similar manner to the EOM without damping. Again, assume the solution:

\[ x(t) = ae^{\lambda t} \]  \hspace{1cm} (79)

de where \( a \) and \( t \) are nonzeros constants that need to be determined. Using successive differentiation, we get:

\[ \dot{x}(t) = \lambda ae^{\lambda t} \]  \hspace{1cm} (80)

and

\[ \ddot{x}(t) = \lambda^2 ae^{\lambda t} \]  \hspace{1cm} (81)

therefore, \( m\ddot{x} + c\dot{x} + kx = 0 \) becomes:

\[ m\lambda^2 ae^{\lambda t} + c\lambda ae^{\lambda t} + kae^{\lambda t} = 0 \]  \hspace{1cm} (82)

Now we divide by \( ae^{\lambda t} \) to obtain the characteristic equation:

\[ m\lambda^2 + c\lambda + k = 0 \]  \hspace{1cm} (83)
We can do this because $ae^{\lambda t}$ is never zero, therefore, we never divide by zero. The quadratic formula gives us:

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} = \frac{-c \pm \frac{1}{2m} \sqrt{c^2 - 4km}}{2m}$$ \hspace{1cm} (84)$$

Some key points from this equation:

- The $\pm$ tells us there are two solutions to this problem
- if $c^2 - 4km < 0$, system is Underdamped, solutions are complex conjugate pairs
- if $c^2 - 4km = 0$, system is critically damped, solutions are equal negative real numbers
- if $c^2 - 4km > 0$, system is Overdamped, solutions are distinct negative real numbers

From this, we can see that $c^2 - 4km = 0$ is a special value, let us define a value for $c$ that will give us this critical damping number. We will call it the **critical damping coefficient** ($c_{cr}$). So setting the equation as:

$$c_{cr}^2 - 4km = 0$$ \hspace{1cm} (85)$$

giving us:

$$c_{cr}^2 = 4km$$ \hspace{1cm} (86)$$

next we can derive the function:

$$c_{cr} = 2\sqrt{km} = 2\left(\frac{\sqrt{m}}{\sqrt{m}}\right)\sqrt{km} = 2m\omega_n$$ \hspace{1cm} (87)$$

remember that $\omega_n = \sqrt{\frac{k}{m}}$ for an undamped system. Next, we generate a non-dimensional number ($\zeta$), pounced ‘zeta’ that will allow us to distinguish between different types of damping. $\zeta$ is called the **critical damping ratio**.

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega_n}$$ \hspace{1cm} (88)$$

Now if we put the $\zeta$ back into the characteristic equation and resolve using the quadratic equation we get:

$$\lambda_{1,2} = -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$$ \hspace{1cm} (89)$$
From this equation it become clear that $\zeta$ determines whether the roots are complex or real, this in turn determines the nature of the response of the structure. Listing our possible responses we get: For each damping case, we will have a different solution to the problem.

<table>
<thead>
<tr>
<th>damping case</th>
<th>critical damping ratio</th>
<th>radicand</th>
<th>solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>under damped</td>
<td>$0 &lt; \zeta &lt; 1$</td>
<td>$c^2 - 4km &lt; 0$</td>
<td>complex conjugate pairs</td>
</tr>
<tr>
<td>critically damped</td>
<td>$\zeta = 1$</td>
<td>$c^2 - 4km = 0$</td>
<td>equal negative real numbers</td>
</tr>
<tr>
<td>over damped</td>
<td>$1 &lt; \zeta$</td>
<td>$c^2 - 4km &gt; 0$</td>
<td>distinct negative real numbers</td>
</tr>
</tbody>
</table>

### 2.3.2 Modeling Underdamped Motion

In the case that $0 < \zeta < 1$, a complex conjugate pair of roots are the solutions to the characteristic equation after pulling out a $\sqrt{-1}$:

$$\lambda_1 = -\zeta \omega_n + \omega_n \sqrt{1 - \zeta^2} j$$

and:

$$\lambda_2 = -\zeta \omega_n - \omega_n \sqrt{1 - \zeta^2} j$$

Where the $j$ is pulled out because:

$$\sqrt{1 - \zeta^2} j = \sqrt{(1 - \zeta^2)(-1)} = \sqrt{\zeta^2 - 1}$$

Next, let us “arbitrarily” define:

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

where $\omega_d$ is the damped natural frequency. Therefore, the equations become:

$$\lambda_1 = -\zeta \omega_n + \omega_d j$$

and:

$$\lambda_2 = -\zeta \omega_n - \omega_d j$$

Again, we have two solutions to a linear problem, so we can combine these into one solution and insert $\lambda$ into the assumed solution $ae^{\lambda t}$ to obtain:

$$x(t) = a_1 e^{-\zeta \omega_n t + \omega_d t j} + a_2 e^{-\zeta \omega_n t - \omega_d t j}$$
where $a_1$ and $a_2$ are complex valued constants. This can now be simplified into:

$$x(t) = e^{-\zeta \omega_n t} (a_1 e^{\omega_d t j} + a_2 e^{-\omega_d t j})$$  \hspace{1cm} (97)

Using Euler’s equations, (same as before) and choosing:

$$A_1 = (a_1 - a_2) j$$ \hspace{1cm} (98)

and

$$A_2 = (a_1 + a_2)$$ \hspace{1cm} (99)

The **general form** of this solution is then:

$$x(t) = e^{-\zeta \omega_n t} (A_1 \sin(\omega_d t) + A_2 \cos(\omega_d t))$$ \hspace{1cm} (100)

Recall that for undamped 1-DOF systems we showed

$$x(t) = A \sin(\omega_n t + \phi) = A_1 \sin(\omega_n t) + A_2 \cos(\omega_n t)$$ \hspace{1cm} (101)

As $e^{-\zeta \omega_n t}$ accounts for the damping, our current solution becomes:

$$x(t) = A e^{-\zeta \omega_n t} \sin(\omega_d t + \phi)$$ \hspace{1cm} (102)

Now that we have $x$ and $\dot{x}$, we can solve for the boundary conditions $x_0$ and $v_0$ by setting $t = 0$, we get:

$$x(0) = x_0 = A \sin(\phi)$$ \hspace{1cm} (103)

and taking the directive of $x(t)$ using the product rule $(fg)' = f'g + fg'$, we get:

$$\dot{x}(t) = -\zeta \omega_n A e^{-\zeta \omega_n t} \sin(\omega_d t + \phi) + A e^{-\zeta \omega_n t} \omega_d \cos(\omega_d t + \phi)$$ \hspace{1cm} (104)

$$\dot{x}(0) = v_0 = -\zeta \omega_n A \sin(\phi) + A \omega_d \cos(\phi)$$ \hspace{1cm} (105)

a simplification can be made to the prior equation by letting $A = x_0 / \sin(\phi)$. This gives us the equation:

$$\dot{x}(0) = v_0 = -\zeta \omega_n \left( \frac{x_0}{\sin(\phi)} \right) \sin(\phi) + \left( \frac{x_0}{\sin(\phi)} \right) \omega_d \cos(\phi)$$ \hspace{1cm} (106)

that can be simplified to:

$$\dot{x}(0) = v_0 = -\zeta \omega_n x_0 + x_0 \omega_d \cot(\phi)$$ \hspace{1cm} (107)
The above equation related \( v_0 \) to \( \phi \) using terms that are known for a giving system \((\zeta, \omega_n, x_0, \text{ and } \omega_d)\). Therefore, this expression can be used to solve for \( \phi \):

\[
\cot(\phi) = \frac{v_0 + \zeta \omega_n x_0}{x_0 \omega_d} \tag{108}
\]

and as the \( \tan(\phi) = 1/\cot(\phi) \):

\[
\phi = \tan^{-1}\left(\frac{x_0 \omega_d}{v_0 + \zeta \omega_n x_0}\right) \tag{109}
\]

Thereafter, we can solve for \( A \) considering the fact that we sent \( A = x_0/\sin(\phi) \). Using the trigonometric relationship between expressed in equation 108 and visualized in figure 16:

\[
\sqrt{(v_0 + \zeta \omega_n x_0)^2 + (x_0 \omega_d)^2}
\]

\[x_0 \omega_d \]

\[v_0 + \zeta \omega_n x_0 \]

Figure 16: Trigonometric relationship between the initial conditions \((x_0 \text{ and } v_0)\), amplitude \( A \), and phase \( \phi \) for underdamped motion of a 1-DOF system.

we show that \( \sin(\phi) \) can be expressed as:

\[
\sin(\phi) = \frac{x_0 \omega_d}{\sqrt{(v_0 + \zeta \omega_n x_0)^2 + (x_0 \omega_d)^2}} \tag{110}
\]

and applying \( A = x_0/\sin(\phi) \) we get:

\[
A = \frac{\sqrt{(v_0 + \zeta \omega_n x_0)^2 + (x_0 \omega_d)^2}}{\omega_d} = \sqrt{x_0^2 + \left(\frac{v_0 + \zeta \omega_n x_0}{\omega_d}\right)^2} \tag{111}
\]

Finally, collecting all of our important equations:

- Critical damping coefficient: \( c_{cr} = 2\sqrt{km} = 2m\omega_n \)
• Damping ratio: $\zeta = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega_n}$

• Damped natural frequency: $\omega_d = \omega_n \sqrt{1 - \zeta^2}$

• Solution for underdamped system: $x(t) = Ae^{-\zeta \omega_n t} \sin(\omega_d t + \phi)$, where:

\[
A = \frac{\sqrt{(v_0 + \zeta \omega_n x_0)^2 + (x_0 \omega_d)^2}}{\omega_d}
\]

\[
\phi = \tan^{-1}\left(\frac{x_0 \omega_d}{v_0 + \zeta \omega_n x_0}\right)
\]

Figure 17: Four example responses for an underdamped 1-DOF system ($\zeta = 0.142$) with various initial conditions.

**Example 2.5** Consider the following 1-DOF system, where $k = 857.8$ N/m, $c = 7.8$ kg/s, and $m = 49.2 \times 10^{-3}$ kg, calculate the damped frequency in rad/s and Hz. What damping case is this system?
Figure 18: 1-DOF spring-dashpot-mass system.

Solution:

Calculate the undamped frequency:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{857.8}{49.2 \times 10^{-3}}} = 132 \text{ rad/s} \quad (112)$$

The systems critical damping value:

$$c_{cr} = 2\sqrt{km} = 2\sqrt{k} = 857.8 \cdot 49.2 \times 10^{-3} = 12.993 \text{ kg/s} \quad (113)$$

And the critical damping ratio:

$$\zeta = \frac{c}{c_{cr}} = \frac{7.8}{12.993} = 0.600 \quad (114)$$

This can also be expressed as 60% damped, this is a underdamped system, and the system will oscillate. Now we can calculate the damped frequency:

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = \omega_n \sqrt{1 - 0.600^2} = 105.6 \text{ rad/s} \quad (115)$$

Therefore, the system oscillates at 105.6 rad/sec or 16.81 Hz

Example 2.6 For a damped one DOF system where $m$, $c$, and $k$ are known to be $m = 1$ kg, $c = 2$ kg/s, and $k = 10$ N/m. Calculate the value of $\zeta$ and $\omega_n$. Is the system overdamped, underdamped, or critically damped?

Solution:
The natural frequency is calculated as

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10}{1}} = 3.16 \text{ rad/s} \quad (116)$$

The damping can be calculated as:

$$\zeta = \frac{c}{2\omega_nm} = \frac{2}{2\left(\sqrt{\frac{10}{1}}\right)}(1) = \frac{1}{\sqrt{10}} = 0.316 \quad (117)$$

So the damped natural frequency is equal to:

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} = \sqrt{10}\sqrt{1 - \left(\frac{1}{\sqrt{10}}\right)^2} = 0.3 \text{ rad/s} \quad (118)$$

As $0 < \zeta < 1$ the system is underdamped.

**Example 2.7** For the following industrial device consisting of a mass isolated from its fixtures by two rubber dampers and a offset spring provide an estimate of the system’s damped natural frequency in the vertical direction. Assume the the rubber dampers add damping and only negligible stiffness to the system and that the spring is long enough such that the angles remain constant.

![Diagram](image.png)

Figure 19: Industrial device (mass) connected to a fixed point with a rubber damper and spring at an angle.

**Solution:**

First and foremost, we need to develop a mass-spring-dashpot representa-
tion of the system. The is presented in what follows:

![Diagram of mass-spring-dashpot system](image)

Figure 20: Mass-spring-dashpot representation of the industrial system represented if figure 19.

where the damping in the vertical direction provided by the rubber damper is modeled a dashpot in the vertical direction. As we only want an estimate of the frequency, the assumption that the is small and as such $\alpha$ of the displaced state is equal $\alpha$ of the equilibrium state. This leads to the FBD for the equilibrium and displaced states:

<table>
<thead>
<tr>
<th>equilibrium position</th>
<th>displaced position “x”</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Equilibrium Diagram" /></td>
<td><img src="image" alt="Displaced Diagram" /></td>
</tr>
</tbody>
</table>

The equation for the equilibrium state is:

$$ + \sum F_x = mg - k\delta \cos(\alpha) = 0 $$

and in the displaced state:

$$ + \sum F_x = mg - c\ddot{x} - k\cos(\alpha)(\Delta l + \delta) $$
Applying Newton’s second law and combining these equations yields:

\[ m\ddot{x} + c\dot{x} + k\Delta l\cos(\alpha) = 0 \]

Looking at the triangles formed by the dashpot and spring it can be shown that:

\[ \cos(\alpha) = h/l = x/\Delta l \]

As we assumed the displacement is small and \( \alpha \) remains unchanged. Therefore the prior equation becomes:

\[ m\ddot{x} + c\dot{x} + k\Delta l \frac{x}{\Delta l} = 0 \]

This simplifies to the “normal” EOM for a 1-DOF system:

\[ m\ddot{x} + c\dot{x} + kx = 0 \]

Therefore, once the values for the system are measured the system’s damped natural frequency in the vertical direction can be estimated as

\[ \omega_d = \omega_n \sqrt{1 - \zeta^2} \]

2.3.3 Modeling Overdamped Motion

In the case of overdamped systems, \( 1 < \zeta \), the solutions for \( \lambda \) are distinct real roots that are written as:

\[ \lambda_1 = -\zeta \omega_n - \omega_n \sqrt{\zeta^2 - 1} \quad (119) \]

and:

\[ \lambda_2 = -\zeta \omega_n + \omega_n \sqrt{\zeta^2 - 1} \quad (120) \]

The solution for the EOM using the assumed solution then becomes:

\[ x(t) = e^{-\zeta \omega_n t} (a_1 e^{-\omega_n t \sqrt{\zeta^2 - 1}} + a_2 e^{+\omega_n t \sqrt{\zeta^2 - 1}}) \quad (121) \]
This equation represents a non-oscillating response of the system. Again, \( a_1 \) and \( a_2 \) are solved for using known boundary conditions \( x_0 \) and \( v_0 \) such that:

\[
a_1 = \frac{-v_0 + \left( -\zeta + \sqrt{\zeta^2 - 1} \right) \omega_n x_0}{2\omega_n \sqrt{\zeta^2 - 1}}
\]

(122)

\[
a_2 = \frac{v_0 + \left( \zeta + \sqrt{\zeta^2 - 1} \right) \omega_n x_0}{2\omega_n \sqrt{\zeta^2 - 1}}
\]

(123)

Typical responses for a overdamped system with various initial conditions are shown below:

![Figure 21: Four example responses for an over damped 1-DOF system (\( \zeta = 2.371 \)) with various initial conditions.](image)

Figure 21: Four example responses for an over damped 1-DOF system (\( \zeta = 2.371 \)) with various initial conditions.

2.3.4 Modeling critically damped motion

In the case of critically damped systems, \( \zeta = 1 \), the solutions for \( \lambda \) will be equal negative real numbers, therefore from before:

\[
\lambda_{1,2} = -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1}
\]

(124)

We get:

\[
\lambda_1 = \lambda_2 = -\omega_n
\]

(125)

Because both solutions (\( a_1 \) and \( a_2 \)) are the same, we multiply the second solution by \( t \) so the solution for a critically damped system is in the same form as before.
The solution for the EOM using the assumed solution then becomes:

\[ x(t) = a_1 e^{-\omega_n t} + a_2 t e^{-\omega_n t} \quad (126) \]

This simplifies into:

\[ x(t) = (a_1 + a_2 t) e^{-\omega_n t} \quad (127) \]

This equation represents a non-oscillating response of the system. Again, \( a_1 \) and \( a_2 \) are solved for using known boundary conditions \( x_0 \) and \( v_0 \) such that:

\[ a_1 = x_0 \quad (128) \]
\[ a_2 = v_0 + \omega_n x_0 \quad (129) \]

### 2.3.5 Standard Form of the EOM

The EOM for a damped 1-DOF system is written in a “standard form” in which the effect of the damping ratio and natural frequencies are more obvious. To get to the standard form, the normal form of the EOM:

\[ m \ddot{x} + c \dot{x} + kx = 0 \quad (130) \]

is divided by what the constant terms associated with the acceleration term. In this example, this is \( m \). Dividing every term by \( m \) yields:

\[ \ddot{x} + \frac{c}{m} \dot{x} + \frac{k}{m} x = 0 \quad (131) \]

Numerical manipulations can be undertaken to get the coefficients of the velocity and displacement terms into coefficients that more clearly express the characteristics of the vibrating system:

\[ \ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = 0 \quad (132) \]

### Example 2.8

A engine valve assembly depicted in figure 22 where \( J \) is the inertia caused by the right-hand side of the rocker arm. Derive an analytical solution for the natural frequency of the rocker arm. Use the assumptions \( \sin(\theta) = \theta \) and \( \cos(\theta) = 1 \).
Figure 22: Rocker arm assembly of an internal combustion engine showing: (a) a diagram of the system and; (b) the FBD of the system.

**Solution:**

Taking the sum of the moments about \( O \) and considering the inertia caused by the right-hand side of the rocker arm, \( J \), the FBDs can be written as:

- **equilibrium position**
- **displaced position “x”**

The equation for the equilibrium state is:

\[
\zeta + \sum M_o = mgl - kl^2 \delta = 0
\]

and in the displaced state:

\[
\zeta + \sum M_o = mgl - kl^2 \delta - kl^2 \theta - cl^2 \dot{\theta} = 0
\]
Applying Newton’s second law and combining these equations yields:

\[(J + ml^2)\ddot{\theta} + cl^2 \dot{\theta} + kl^2 \theta = 0\]  
(133)

Therefore, the standard form of the EOM is:

\[\ddot{\theta} + \frac{cl^2}{J + ml^2} \dot{\theta} + \frac{kl^2}{J + ml^2} \theta = 0\]  
(134)

Results in the following analytical solution for the natural frequency:

\[\omega_n = \sqrt{\frac{kl^2}{J + ml^2}} \text{ rad/s}\]  
(135)

### 2.4 Logarithmic decrement

For a vibrating system, the mass \(m\) and stiffness \(k\) can be measured using scales and static deflection tests. However, the damping coefficient \(c\) is a more difficult quantity to determine. From \(k\) and \(m\) we can compute the natural frequency \(\omega_n\) and the critical damping coefficient \(c_{cr}\). Therefore, knowing that the critical damping ratio \(\zeta\) is defined as:

\[\zeta = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega_n}\]  
(136)

if we calculate \(\zeta\), we can obtain \(c\) for the system of interest. This is made possible because \(c_{cr}\) can be calculated from \(k\) and \(m\). Observing the temporal response for the underdamped system,
we mark three points of maximum amplitude, $x_1$, $x_2$, and $x_3$ that happen at $t_1$, $t_2$, and $t_3$, respectively. Considering displacement values for the first two points $x_1$ and $x_2$, separated by a complete period ($T$). Knowing that one cycle is $2\pi$, the time period for this complete cycle is given by:

$$t_2 - t_1 = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1 - \zeta^2}}$$  \hspace{1cm} (137)

where $\omega_d$ is the damped natural frequency. This is the time period ($T$) of damped oscillations. If we derive an equation for the values of the peaks, also called the envelope of maximum values, we get:

$$x_{\text{peaks}} = Ae^{-\zeta \omega_n t}$$  \hspace{1cm} (138)

Knowing that the system is underdamped, $A$ can be solved for using the initial conditions $x_0$ and $v_0$, therefore:

$$A = \frac{\sqrt{(v_0 + \zeta \omega_n x_0)^2 + (x_0 \omega_d)^2}}{\omega_d}$$  \hspace{1cm} (139)

In terms of $t_1$ and $t_2$, we can express the displacement at these times as:

$$x_1 = Ae^{-\zeta \omega_n t_1}$$  \hspace{1cm} (140)
and

\[ x_2 = Ae^{-\zeta \omega_n t_2} \]  \hspace{1cm} (141)

therefore:

\[ \frac{x_1}{x_2} = \frac{e^{-\zeta \omega_n t_1}}{e^{-\zeta \omega_n t_2}} = e^{\zeta \omega_n (t_2 - t_1)} \]  \hspace{1cm} (142)

However, from before we know that \( t_2 - t_1 = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1 - \zeta^2}} \). Therefore, we can express this last equation as:

\[ \frac{x_1}{x_2} = e^{\left( \frac{2\pi \zeta}{\sqrt{1 - \zeta^2}} \right)} \]  \hspace{1cm} (143)

Next, we take the natural log of both sides to get the logarithmic decrement, denoted by \( \delta \):

\[ \delta = \ln \left( \frac{x_1}{x_2} \right) = \ln \left( \frac{x(t_1)}{x(t_1 + T)} \right) = \frac{2\pi \zeta}{\sqrt{1 - \zeta^2}} \]  \hspace{1cm} (144)

This shows us that the ratio of any two successive amplitudes for an underdamped system, vibrating freely, is constant and is a function of the damping only. Sometimes, in experiments, it is more convenient/accurate to measure the amplitudes after say “\( n \)” peaks rather than two successive peaks (because if the damping is very small, the difference between the successive peaks may not be significant). The logarithmic decrement can then be given by the equation

\[ \delta = \frac{1}{n} \ln \left( \frac{x_1}{x_{n+1}} \right) = \frac{1}{n} \ln \left( \frac{x(t_1)}{x(t_1 + nT)} \right) = \frac{2\pi \zeta}{\sqrt{1 - \zeta^2}} \]  \hspace{1cm} (145)

Once we use the experimental data to obtain \( \delta \), and knowing that:

\[ \delta = \frac{2\pi \zeta}{\sqrt{1 - \zeta^2}} \]  \hspace{1cm} (146)

we can calculate the value of \( \zeta \):

\[ \zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \]  \hspace{1cm} (147)

Therefore, having \( \zeta \) we can solve for the coefficient of damping, \( c \), as:

\[ c = \zeta 2\sqrt{km} \]  \hspace{1cm} (148)
Example 2.9 Calculate the damping coefficient for the system with the measured amplitude as expressed below given that \( m = 3 \) kg and \( k = 43 \) N/m. Use \( t_1 = 1 \) sec, and \( t_{n+1} = t_4 = 6 \) sec.

![Figure 24: Response from an experimental system with noise.](image)

**Solution:**

First, from the plot we can determine that \( x_1 = -9.5 \) mm and \( x_3 = -1.8 \) mm where \( n = 3 \). Thereafter, we can solve for \( \delta \):

\[
\delta = \frac{1}{3} \ln \left( \frac{x_1}{x_4} \right) = \frac{1}{3} \ln \left( \frac{-9.5}{-1.8} \right) = 0.554
\]  

(149)

Next, we can calculate \( \zeta \), as:

\[
\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \frac{0.554}{\sqrt{4\pi^2 + 0.554^2}} = 0.0879
\]

(150)

And lastly:

\[
c = \zeta \cdot 2\sqrt{km} = 0.0879 \cdot 2\sqrt{43 \cdot 3} = 2.0 \text{ kg/s}
\]

(151)

Example 2.10 The free response of a 1000-kg automobile with a stiffness of \( k = 400,000 \) N/m is observed to be underdamped. Modeling the automobile as a single-degree-of-freedom oscillation in the vertical direction, as annotated
in figure 7, determine the damping coefficient if the displacement at \( t_1 \) is measured to be 2 cm and 0.22 cm at \( t_2 \).

**Solution:**

Knowing \( x_1 = 2 \) cm and \( x_2 = 0.22 \) cm and \( t_2 = T + t_1 \), therefore:

\[
\delta = \ln \frac{x_1}{x_2} = \ln \frac{2}{0.22} = 2.207 \tag{152}
\]

and:

\[
\zeta = \left( \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \right) = \left( \frac{2.207}{\sqrt{4\pi^2 + 2.207^2}} \right) = 0.331 \tag{153}
\]

therefore, we can obtain the damping coefficient as

\[
c = 2\zeta \sqrt{km} = 2(0.331)\sqrt{400,000 \cdot 1,000} = 13,256 \text{ kg/s} \tag{154}
\]
3 Forced Vibrations

Mechanical systems are often subjected to an external loading. For example, a piston in an engine if forced up and down by a crankshaft or a seat in an airplane may vibrate due to movement of the jet engines transmitted through the aircraft structure.

3.1 Harmonic Excitations of Undamped Systems

Investigating a single-degree of freedom system for a harmonic input is useful as it can be solved mathematically with straightforward techniques. Consider the system:

where \( F(t) \) is the external force applied to the mass. For simplicity, let us consider a harmonic excitation for \( F(t) \) such that:

\[
F(t) = F_0 \cos(\omega t)
\]  
(1)

note that here, \( \omega \) has no subscript and is the frequency in rad/sec of the driving force. \( F_0 \) represents the magnitude of the applied force. This is often called the input frequency, driving frequency, or forcing frequency. Building the EOM for system in figure 1 yields:

\[
m\ddot{x}(t) + kx(t) = F_0 \cos(\omega t)
\]  
(2)

For convinces we drop the \( (t) \) to make the writing easier can convert this to the standard form by dividing the equation through by \( m \):

\[
\ddot{x} + \omega_n^2 x = f_0 \cos(\omega t)
\]  
(3)
where:

\[ f_0 = \frac{F_0}{m} \]  

(4)

The EOM in this form is a second order, linear nonhomogeneous differential equation. It is nonhomogeneous because there are no terms related to \( x \) on the right-hand side of the equation. One way to solve such such an ODE is to recall that the solution for a nonhomogeneous equation is the sum of the homogeneous and particular solutions.

\[ x = x_h + x_p \]  

(5)

again, noting that this is a temporal solution where “(t)” is implied. First, knowing that the solution is the sum of two parts: 1) oscillations caused by the spring/mass system; and 2) vibrations caused by the forcing function. The oscillations caused by the spring/mass system will form the homogeneous while the vibrations caused by the forcing function will form the particular solution. As we know the solution for oscillations caused by the spring/mass system form our prior investigation of unforced system we set the equation for the homogeneous solution to be:

\[ x_h = A \sin(\omega_n t + \phi) \]  

(6)

Next, we will denote the particular solution as \( x_p \).\( x_p \) can be determined by assuming that it is in form of the forcing function, therefore:

\[ f_0 \cos(\omega t) \]  

(7)

becomes:

\[ x_p = X \cos(\omega t) \]  

(8)

where, \( x_p \) is the particular solution and \( X \) is the amplitude of the forced response. Our total solution for the harmonic excitations of undamped systems now becomes:

\[ x(t) = A \sin(\omega_n t + \phi) + X \cos(\omega t) \]  

(9)

This approach, of assuming that \( x_p = X \cos(\omega t) \), in order to determine the particular solution is called the **method of undetermined coefficients**. To calculate \( X \), first we take the equations for \( x_p \) and \( \ddot{x}_p \):

\[ x_p = X \cos(\omega t) \]  

(10)

\[ \ddot{x}_p = -\omega^2 X \cos(\omega t) \]  

(11)
and substituting these into the equation of motion in standard from yields:

\[-\omega^2 X \cos(\omega t) + \omega_n^2 X \cos(\omega t) = f_0 \cos(\omega t)\]  \hspace{1cm} (12)

As long as \(\cos(\omega t) \neq 0\), solving for \(X\) yields:

\[X = \frac{f_0}{\omega_n^2 - \omega^2}\]  \hspace{1cm} (13)

Therefore, as long as \(\omega_n \neq \omega\), the particular solution can take the form:

\[x_p = \frac{f_0}{\omega_n^2 - \omega^2} \cos(\omega t)\]  \hspace{1cm} (14)

This then expands to the total form:

\[x(t) = A \sin(\omega_n t + \phi) + \frac{f_0}{\omega_n^2 - \omega^2} \cos(\omega t)\]  \hspace{1cm} (15)

Expanding this to the general form for the homogeneous solution obtains the equation:

\[x(t) = A_1 \sin(\omega_n t) + A_2 \cos(\omega_n t) + \frac{f_0}{\omega_n^2 - \omega^2} \cos(\omega t)\]  \hspace{1cm} (16)

As before, we need to determine the values for the coefficients \(A_1\) and \(A_2\) by enforcing the initial conditions \(x_0\) and \(v_0\). Setting the time to zero \((t = 0)\) and solving the initial displacement leads to:

\[x(0) = x_0 = A_2 + \frac{f_0}{\omega_n^2 - \omega^2}\]  \hspace{1cm} (17)

or:

\[A_2 = x_0 - \frac{f_0}{\omega_n^2 - \omega^2}\]  \hspace{1cm} (18)

again, solving the equation in terms of velocity:

\[\dot{x}(t) = A_1 \omega_n \cos(\omega_n t) - A_2 \omega_n \sin(\omega_n t) - \omega \frac{f_0}{\omega_n^2 - \omega^2} \sin(\omega t)\]  \hspace{1cm} (19)

and solving for the initial velocity at \(t = 0\):

\[\dot{x}(0) = v_0 = A_1 \omega_n\]  \hspace{1cm} (20)

or:

\[A_1 = \frac{v_0}{\omega_n}\]  \hspace{1cm} (21)
Therefore, combining the equations we get:

\[ x(t) = \left( \frac{v_0}{\omega_n} \right) \sin(\omega_n t) + \left( x_0 - \frac{f_0}{\omega_n^2 - \omega^2} \right) \cos(\omega_n t) + \frac{f_0}{\omega_n^2 - \omega^2} \cos(\omega t) \]  (22)

As before, we can relate \( A_1 \) and \( A_2 \) to each other through the basic trigonometric identities. This yields,

\[ x(t) = A \sin(\omega_n t + \phi) + X \cos(\omega t) \]  (23)

\[ A = \sqrt{\left( \frac{v_0}{\omega_n} \right)^2 + (x_0 - X)^2} \]  (24)

\[ \phi = \tan^{-1} \left( \frac{\omega_n (x_0 - X)}{v_0} \right) \]  (25)

\[ X = \frac{f_0}{\omega_n^2 - \omega^2} \]  (26)

**Example 3.1** For a the 1-DOF system:

![1-DOF spring-mass system subjected to an external force](image)

with \( k = 10 \text{ N/m} \), \( m = 2.5 \text{ kg} \), \( \omega = 4 \text{ rad/sec} \), \( F_0 = 0.1 \text{ kN} \), \( x_0 = 1 \text{ mm} \), and \( v_0 = 0 \text{ mm} \) plot the temporal responses of system considering the free-vibration case and the excited case. Plot these on a single plot to compare the responses.

**Solution:** The free-vibration response can be plotted using the expression:

\[ x(t) = x_0 \cos(\omega_n t) + \frac{v_0}{\omega_n} \sin(\omega_n t) \]  (27)

while the force vibration is expressed using:

\[ x(t) = \left( \frac{v_0}{\omega_n} \right) \sin(\omega_n t) + \left( x_0 - \frac{f_0}{\omega_n^2 - \omega^2} \right) \cos(\omega_n t) + \frac{f_0}{\omega_n^2 - \omega^2} \cos(\omega t) \]  (28)
These temporal responses are plotted as (Note that the forcing function uses the axis on the right):

![Comparison of the temporal response for a 1-DOF system; expressing how the forcing function changes the vibrational temporal response of the system.](image)

Figure 3: Comparison of the temporal response for a 1-DOF system; expressing how the forcing function changes the vibrational temporal response of the system.

### 3.2 Harmonic Resonance

Recall that our solution from before assumed that $\omega_n \neq \omega$, however, if $\omega_n = \omega$ then the system will develop the phenomenon of resonance. Mathematically, this means the amplitude of the vibrations becomes unbounded. The prior choice of $X \cos(\omega t)$ for the particular solution fails as it is also a solution for a homogeneous equation. Therefore, a new particular solution is needed for the case where $\omega_n = \omega$. This new particular solution can be written as:

$$x_p(t) = tX \sin(\omega t)$$  \hspace{1cm} (29)

Substituting this into the EOM of the system in standard form equation (from Boyce and DiPrima (1997)) and solving for $X$ yields:

$$x_p(t) = \frac{f_0}{2\omega} t \sin(\omega t)$$  \hspace{1cm} (30)

thus, the total solution can now be written as:

$$x(t) = A_1 \sin(\omega t) + A_2 \cos(\omega t) + \frac{f_0}{2\omega} t \sin(\omega t)$$  \hspace{1cm} (31)
Note that $\omega_n = \omega$, therefore, the frequencies are all in terms of the driving frequency $\omega$. Again, evaluating the solution at $t = 0$ for the initial conditions $x_0$ and $v_0$ yields:

$$x(t) = \left(\frac{v_0}{\omega}\right)\sin(\omega t) + x_0\cos(\omega t) + \frac{f_0}{2\omega}t\sin(\omega t)$$

(32)

Where the first two terms account for the oscillations while the third terms accounts for the continued increase of the maximum amplitude. The following plot shows the forced response of a spring-mass system driven harmonically at its natural frequency.

**Example 3.2** Compute solutions for the homogeneous and particular solution separately, than compute the total response of a spring-mass system with the following values: $k = 1000$ N/m, $m = 10$ kg, subject to a harmonic force of magnitude $F_0 = 100$ N and frequency of $8.162$ rad/s, and initial conditions given by $x_0 = 0$ m and $v_0 = 0$ m/s. Plot the response.

![Diagram of spring-mass system](image)
Solution: First, make sure that the system is not in resonance. Calculating that \( \omega_n = \sqrt{1000/10} = 10 \) shows us that \( \omega_n \neq \omega \). Next knowing that \( f_0 = F_0/m = 10 \) we can find the homogeneous and particular solutions as:

\[
x_h(t) = A \sin(\omega_n t + \phi) \tag{33}
\]

\[
x_p(t) = X \cos(\omega t) \tag{34}
\]

also:

\[
x(t) = x_h(t) + x_p(t) \tag{35}
\]

where:

\[
A = \sqrt{\left( \frac{v_0}{\omega_n} \right)^2 + (x_0 - X)^2} \tag{36}
\]

\[
\phi = \tan^{-1} \left( \frac{\omega_n (x_0 - X)}{v_0} \right) \tag{37}
\]

\[
X = \frac{f_0}{\omega_n^2 - \omega^2} \tag{38}
\]

This leads to the following results.

**Example 3.3** Considering the following system, write the equation of motion and calculate the response assuming a) that the system is initially at rest, and b) that the system has an initial displacement of 0.005 m. Use \( k = 2000 \text{ N/m} \),
\( m = 100 \text{ kg}, \quad F(t) = 10\sin(10t) \text{ N.} \)

![1-DOF spring-mass system](image)

Figure 5: 1-DOF spring-mass system subjected to an external force \( F(t) \).

**Solution:** The equation of motion is

\[
m\ddot{x} + kx = 10\sin(10t)
\]  \hspace{1cm} (39)

or in standard form:

\[
\ddot{x} + \omega_n^2 x = f_0 \sin(\omega t)
\]  \hspace{1cm} (40)

Note that the forcing function is in terms of \( \sin \), not \( \cos \) as before, so we will have to resolve for the constants \( A_1 \) and \( A_2 \). Again, setting the particular solution to \( x_p = X \sin(\omega t) \) and solving for \( X \) as before yields:

\[
x(t) = A_1 \sin(\omega_n t) + A_2 \cos(\omega_n t) + \frac{f_0}{\omega_n^2 - \omega^2} \sin(\omega t)
\]  \hspace{1cm} (41)

Now we can solve for \( A_1 \) and \( A_2 \) by setting the initial conditions \( x_0 \) and \( v_0 \) to \( t = 0 \). First, setting \( t = 0 \) in the equation for \( x(t) \) yields:

\[
A_2 = x_0
\]  \hspace{1cm} (42)

Then, a function for the velocity of the system is obtained:

\[
\dot{x}(t) = v_0 = A_1 \omega_n \cos(\omega_n t) - A_2 \omega_n \sin(\omega_n t) + \omega \frac{f_0}{\omega_n^2 - \omega^2} \cos(\omega t)
\]  \hspace{1cm} (43)

This allows us to obtains:

\[
A_1 = \frac{v_0 - \frac{\omega}{\omega_n} \cdot f_0}{\omega_n^2 - \omega^2}
\]  \hspace{1cm} (44)
at $t = 0$. These lead to the full equation for the general solution:

$$x(t) = \left( \frac{v_0}{\omega_n} - \frac{\omega}{\omega_n} \cdot \frac{f_0}{\omega_n^2 - \omega^2} \right) \sin(\omega_n t) + x_0 \cos(\omega_n t) + \frac{f_0}{\omega_n^2 - \omega^2} \sin(\omega t) \quad (45)$$

Also, knowing:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{20 \text{ rad/sec}} = 4.472 \text{ rad/sec} \quad (46)$$

and

$$f_o = \frac{F_0}{m} = \frac{F_0}{m} = 0.1 \text{ N/kg} \quad (47)$$

a) using the initial conditions $x_0 = 0 \text{ m}$ and $v_0 = 0 \text{ m/s}$ and the general expression obtained above:

$$x(t) = \left( 0 - \frac{10}{\sqrt{20}} \cdot \frac{0.1}{20 - 10^2} \right) \sin(\sqrt{20} t) + 0 + \frac{0.1}{20 - 10^2} \sin(10 t) \quad (48)$$

b) using the initial conditions $x_0 = 0.005 \text{ m}$ and $v_0 = 0 \text{ m/s}$ and the general expression obtained above:

$$x(t) = \left( 0 - \frac{10}{\sqrt{20}} \cdot \frac{0.1}{20 - 10^2} \right) \sin(\sqrt{20} t) + 0.05 \cos(\sqrt{20} t) + \frac{0.1}{20 - 10^2} \sin(10 t) \quad (49)$$
3.3 Harmonic Excitations of Underdamped Systems

Consider the system:

![Diagram of damped 1-DOF system](image)

Figure 6: Damped 1-DOF system with an external force \( F(t) \) applied, showing: (a) the system configuration; and (b) the free body diagram

Again, for simplicity let us consider a harmonic excitation for \( F(t) \) such that:

\[
F(t) = F_0 \cos(\omega t)
\]  

Building the EOM for the above system results in:

\[
m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F_0 \cos(\omega t)
\]  

For convinces we can convert this to the standard form:

\[
\ddot{x}(t) + 2\zeta \omega_n \dot{x}(t) + \omega_n^2 x(t) = f_0 \cos(\omega t)
\]  

again, where:

\[
f_0 = \frac{F_0}{m}
\]  

Recall that one way to solve such an equation is to obtain the sum of the homogeneous and particular solutions.

\[
x(t) = x_h(t) + x_p(t)
\]  

However, now that we have damping force to consider, our particular solution will have to consider this damping. Therefore:

\[
x_p(t) = X \cos(\omega t - \phi_p)
\]  

where \( \phi_p \) represents the phase shift. Note: \( \phi_p \) is represented in other texts as \( \theta \), \( \theta_p \), or even just \( \phi \) but we will use \( \phi_p \) throughout the remainder of this text. Again,
the phase shift is expected because of the effect of the damping force. Now, our total equation is:

\[ x(t) = Ae^{-\zeta \omega_n t} \sin(\omega_d t + \phi) + X \cos(\omega t - \phi_p) \quad (56) \]

We can use the method of undetermined coefficients to obtain \( X \) and \( \phi_p \) for the particular solution. First, considering that we write the particular solution in equivalent form:

\[ x_p(t) = X \cos(\omega t - \phi_p) = A_s \cos(\omega t) + B_s \sin(\omega t) \quad (57) \]

Taking the derivative of the assumed forms of the particular solution yields:

\[ x_p(t) = A_s \cos(\omega t) + B_s \sin(\omega t) \quad (58) \]
\[ \dot{x}_p(t) = -\omega A_s \sin(\omega t) + \omega B_s \cos(\omega t) \quad (59) \]
\[ \ddot{x}_p(t) = -\omega^2 A_s \cos(\omega t) - \omega^2 B_s \sin(\omega t) \quad (60) \]

Recall that the homogeneous and particular solutions are each solutions on their own, therefore, the EOM can be used to describe just the particular solution. Substituting \( x_p, \dot{x}_p, \) and \( \ddot{x}_p \) for \( x, \dot{x}, \) and \( \ddot{x} \) in the EOM in standard form:

\[ \ddot{x}(t) + 2\zeta \omega_n \dot{x}(t) + \omega_n^2 x(t) = f_0 \cos(\omega t) \quad (61) \]

yields:

\[ (-\omega^2 A_s \cos(\omega t) - \omega^2 B_s \sin(\omega t))(t) + 2\zeta \omega_n (\omega A_s \sin(\omega t) + \omega B_s \cos(\omega t))(t) + \]
\[ \omega_n^2 (A_s \cos(\omega t) + B_s \sin(\omega t))(t) = f_0 \cos(\omega t) \quad (62) \]

and rearranging in terms of \( \sin(\omega t) \) and \( \cos(\omega t) \) yields:

\[ (-\omega^2 A_s + 2\zeta \omega_n \omega B_s + \omega_n^2 A_s - f_0) \cos(\omega t) + \]
\[ (-\omega^2 B_s - 2\zeta \omega_n \omega A_s + \omega_n^2 B_s) \sin(\omega t) = 0 \quad (63) \]

From this expression it is clear that there are two special moments in time where \( \cos(\omega t) \) and \( \sin(\omega t) \) equal zero. First, considering that \( t = \pi/(2\omega) \) results in \( \cos(\omega t) = 0, \sin(\omega t) = 1 \) and the equation simplifies to:

\[ (-2\zeta \omega_n \omega) A_s + (\omega_n^2 - \omega^2) B_s = 0 \quad (64) \]

Additionally, at \( t = 0, \sin(\omega t) = 0 \) and \( \cos(\omega t) = 1 \). Therefore, the equation yields

\[ (\omega_n^2 - \omega^2) A_s + (2\zeta \omega_n \omega) B_s = f_0 \quad (65) \]
We can solve two equations for two unknowns. Writing the two linear equations as the singular matrix equation yields:

\[
\begin{bmatrix}
\omega_n^2 - \omega^2 & 2\zeta\omega_n\omega \\
-2\zeta\omega_n\omega & \omega_n^2 - \omega^2
\end{bmatrix}
\begin{bmatrix}
A_s \\
B_s
\end{bmatrix}
= 
\begin{bmatrix}
f_0 \\
0
\end{bmatrix}
\]  
(66)

This can be solved by computing the complex inversing, to give us:

\[
A_s = \frac{(\omega_n^2 - \omega^2)f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}
\]  
(67)

\[
B_s = \frac{2\zeta\omega_n\omega f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}
\]  
(68)

From trigonometric relationships we can see that,

\[
X = \sqrt{A_s^2 + B_s^2}
\]  
(69)

\[
\phi_p = \tan^{-1}\left(\frac{B_s}{A_s}\right)
\]  
(70)

We can now derive values for our particular solution \(x_p\):

\[
X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}
\]  
(71)

\[
\phi_p = \tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right)
\]  
(72)

Now we can build a solution for the particular equation \((x_p)\), therefore, the total solution becomes:

\[
x(t) = x_h(t) + x_p(t)
\]  
(73)

\[
x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_n t + \phi) + X\cos(\omega t - \phi_p)
\]  
(74)

Note for larger values of \(t\), the homogeneous solution approaches zero resulting in the particular solution becoming the total solution. Therefore, the particular solution is sometimes called the steady state response and the homogeneous solution is called the transient response. Solving for the constants \(A\) and \(\phi\) using boundary conditions \((x_0 = 0\) and \(v_0 = 0)\) results a total solution expressed as:

\[
A = \frac{x_0 - X\cos(\phi_p)}{\sin(\phi)}
\]  
(75)
\[ \phi = \tan^{-1} \left( \frac{\omega_d(x_0 - X \cos(\phi_p))}{v_0 + (x_0 - X \cos(\phi_p)) \zeta \omega_n - \omega X \sin(\phi_p)} \right) \]  \hspace{1cm} (76)

Finally, assembling all the terms:

\[ x(t) = Ae^{-\zeta \omega_n t} \sin(\omega_d t + \phi) + X \cos(\omega t - \phi_p) \]  \hspace{1cm} (77)

\[ A = \frac{x_0 - X \cos(\phi_p)}{\sin(\phi)} \]  \hspace{1cm} (78)

\[ \phi = \tan^{-1} \left( \frac{\omega_d(x_0 - X \cos(\phi_p))}{v_0 + (x_0 - X \cos(\phi_p)) \zeta \omega_n - \omega X \sin(\phi_p)} \right) \]  \hspace{1cm} (79)

\[ X = \frac{f_0}{\sqrt{\omega_n^2 - \omega^2}^2 + (2 \zeta \omega_n \omega)^2} \]  \hspace{1cm} (80)

\[ \phi_p = \tan^{-1} \left( \frac{2 \zeta \omega_n \omega}{\omega_n^2 - \omega^2} \right) \]  \hspace{1cm} (81)

**Example 3.4** Consider the damped 1-DOF system below, plot the total, steady state, and transient responses for the following system configurations with no initial conditions. For each configuration, comment on the temporal response and how it differs from the response of the previous configuration.

a) \( k = 100 \text{ N/m}, m = 10 \text{ kg}, c = 10 \text{ kg/s}, F_0 = 1 \text{ N}, \) and \( \omega = 8.162. \)

b) \( k = 100 \text{ N/m}, m = 10 \text{ kg}, c = 10 \text{ kg/s}, F_0 = 3 \text{ N}, \) and \( \omega = 8.162. \)

c) \( k = 100 \text{ N/m}, m = 10 \text{ kg}, c = 10 \text{ kg/s}, F_0 = 3 \text{ N}, \) and \( \omega = 3.162. \)

Figure 7: Damped 1-DOF system with an external force \( (F(t)) \) applied, showing: (a) the system configuration; and (b) the free body diagram
Solution: The total response for the damped 1-DOF system subjected to an external force is modeled using equations 77 through 81 while the transient response consists of the first half of equation 77 and the steady state response consists of the second half of equation 77.

Solution a): Therefore, plotting the temporal responses for configuration a yields:

![Figure 8: Temporal responses for a underdamped system with \( k = 100 \text{ N/m}, m = 10 \text{ kg}, c = 10 \text{ kg/s}, F_0 = 1 \text{ N}, \) and \( \omega = 8.162. \)]

Solution b): Configuration b increases the forcing function \( F_0 \) to 3 N. This results a similar response to configuration a but with a lineally scaled amplitude:
Figure 9: Temporal responses for a underdamped system with $k = 100 \text{ N/m}$, $m = 10 \text{ kg}$, $c = 10 \text{ kg/s}$, $F_0 = 3 \text{ N}$, and $\omega = 8.162$.

**Solution c):** Now, using $\omega = 3.162 \text{ rad/sec}$ we put the system into resonance as $\omega = \omega_n$. However, unlike the undamped system the amplitude of the displacement is not unbounded as the damper absorbs energy from the system. Therefore, after about 7 seconds the systems enters an equilibrium state where any additional increase in amplitude caused by the system entering into resonance is canceled out by the damping in the system as demonstrated in the plot below:
3.4 Frequency Response of Underdamped Systems

From equations 77 through 81 and the figures in example 3.4 we can see that for larger values of \( t \) the transient response dies out while only the steady state response controls the displacement of the total response. This is always true if the system has any significant damping. Therefore, it is often prudent to ignore the transient part and focus only on the steady-state response. Considering the equation for the particular solution:

\[
x_p(t) = X \cos(\omega t - \phi_p)
\]  

and knowing the values for \( X \) and \( \phi_p \):

\[
X = \frac{f_0}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta \omega_n \omega\right)^2}}
\]

\[
\phi_p = \tan^{-1}\left(\frac{2\zeta \omega_n \omega}{\omega_n^2 - \omega^2}\right)
\]

We want to find a way to plot the responses of the system only in terms of terms of the system’s natural and driving frequencies, and its damping. First, we define a frequency ratio as the dimensionless quantity

\[
r = \frac{\omega}{\omega_n}
\]
Another common way to express $r$ is $\beta$. Next, Recall that:

$$X = \frac{f_0}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + (2\zeta \omega_n \omega)^2}} = \frac{F_0}{m \sqrt{\left(\omega_n^2 - \omega^2\right)^2 + (2\zeta \omega_n \omega)^2}}$$

(86)

If we factor out $\omega_n^2$ from the denominator and substitute in $\omega_n^2 = k/m$ and $r = \omega/\omega_n$, we get:

$$X = \frac{F_0}{m \omega_n^2 \sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + (2\zeta \frac{\omega}{\omega_n})^2}} = \frac{F_0}{k \sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

(87)

this becomes:

$$\frac{Xk}{F_0} = \frac{X \omega_n^2}{f_0} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

(88)

in a similar fashion, if we manipulate the equation for $\phi_p$ we can get $\phi_p$ in term of $r$:

$$\phi_p = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$

(89)

If we solve for a few key values of $r$ we can get the following data points. On the board, we can solve for a few different frequency responses for a few different damping coefficients.

<table>
<thead>
<tr>
<th>frequency ratio $(r)$</th>
<th>0</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>1</th>
<th>1.25</th>
<th>1.5</th>
<th>1.75</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\zeta = 0.7$</td>
<td>1.00</td>
<td>1.00</td>
<td>0.97</td>
<td>0.88</td>
<td>0.71</td>
<td>0.54</td>
<td>0.41</td>
<td>0.31</td>
<td>0.24</td>
</tr>
<tr>
<td>$\zeta = 0.5$</td>
<td>1.00</td>
<td>1.03</td>
<td>1.11</td>
<td>1.15</td>
<td>1.00</td>
<td>0.73</td>
<td>0.51</td>
<td>0.37</td>
<td>0.28</td>
</tr>
<tr>
<td>$\zeta = 0.25$</td>
<td>1.00</td>
<td>1.03</td>
<td>1.11</td>
<td>1.15</td>
<td>1.00</td>
<td>0.73</td>
<td>0.51</td>
<td>0.37</td>
<td>0.28</td>
</tr>
<tr>
<td>$\zeta = 0.1$</td>
<td>1.00</td>
<td>1.07</td>
<td>1.32</td>
<td>2.16</td>
<td>5.00</td>
<td>1.62</td>
<td>0.78</td>
<td>0.48</td>
<td>0.33</td>
</tr>
</tbody>
</table>

If we plot the values of the normalize amplitude vs $r$ we obtain:
Figure 11: Normalized amplitude response for frequency ratio ($r$) from 0 to 2 for a variety of critical damping ratios.

And again, if we plot the values of the phase vs $r$ ($\omega/\omega_n$) we obtain:
Figure 12: Phase response for frequency ratio (r) from 0 to 2 for a variety of critical damping ratios.

Note that the dashed black line is there because the phase values after $\pi/2$ need to be adjusted to obtain a continuous plot. An astute observer would notice that the maximum amplitude is not at $\omega = \omega_n$. While resonance is defined as $\omega = \omega_n$, this does not define the point of maximum displacement of the steady state response. Let us solve for the frequency ratio with the maximum displacement. This will happen when

$$d \left( \frac{Xk}{F_0} \right) = 0 \quad (90)$$

We can show that:

$$\left( \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \right) \frac{d}{dr} = 0 \quad (91)$$

when

$$r_{\text{peak}} = \sqrt{1 - 2\zeta^2} = \frac{\omega_p}{\omega_n}, \quad \zeta < 1/\sqrt{2} \quad (92)$$
however, this is only true for under damped system in which $\zeta < 1/\sqrt{2}$. If $\zeta > 1/\sqrt{2}$ than the value in imaginary and the peak value is at $r = 0$. In these cases, the maximum displacement is a function of only $\omega_n$. $\omega_p$ represents the driving frequency that correspond to the maximum amplitude ($\frac{x_k}{F_0}$) and is called the peak frequency, and can be calculated as:

$$\omega_p = \omega_n r_{peak} = \omega_n \sqrt{1 - 2\zeta^2}, \quad \zeta < 1/\sqrt{2} \quad (93)$$

**Example 3.5** Consider the simple spring mass system,

![image](image.png)

Figure 13: Damped 1-DOF spring-mass system subjected to an external force $F(t)$.

where $\omega_n = 132$ rad/sec and $\zeta = 0.0085$. Calculate the displacements of the steady-state response for $\omega=132$ and 125 rad/sec. In both cases, use $f_0 = 10$ N/kg.

**Solution:** From before, we know the solution for the displacement of the particular solution for $\omega=132$ rad/sec is:

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta \omega_n \omega)^2}} = \frac{10}{2(0.0085)(132)^2} = 0.034 \text{ m} \quad (94)$$

while for $\omega=125$ rad/sec $X$ is:

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta \omega_n \omega)^2}} = \frac{10}{\sqrt{(1799)^2 + (280.5)^2}} = 0.005 \text{ m} \quad (95)$$

Therefore, a slight change in the driving frequency (about 5%) results in an 85% change in the amplitude of the steady-state response.

**Example 3.6** The steady state response for an engineered system must not surpass 1 cm, if the system can be modeled as the spring and mass system below, what value of $c$ must be used?
Figure 14: Damped 1-DOF spring-mass system subjected to an external force $F(t)$.

Use $k = 2000$ N/m, $m = 100$ kg, $F(t) = 20 \cos(6.3t)$ N.

**Solution:** The steady state solution is:

$$x_p(t) = X \cos(\omega t - \varphi_p) \tag{96}$$

knowing the amplitude is controlled by $X$:

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2 \zeta \omega_n \omega)^2}} \tag{97}$$

and recalling from the EOM in standard form that $2 \zeta \omega_n = c/m$ we can obtain:

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + \left(\frac{c}{m} \omega\right)^2}} \tag{98}$$

rearranging for $c$ gives:

$$c = m \sqrt{\frac{f_0^2}{\omega^2 X^2 \omega^2} - \frac{(\omega_n^2 - \omega^2)^2}{\omega^2}} = \sqrt{\frac{F_0^2}{\omega^2 X^2 \omega^2} - m^2 \left(\omega_n^2 - \omega^2\right)^2} \tag{99}$$

Therefore, if we set $X = 0.01$ m we can solve the above equation to yield $c = 55.7$ kg/s.

### 3.5 Base Excitation

Often, loading is not applied directly to the mass, but rather the mass of the system is excited when the base of mount that it is attached to is excited. This is called base excitation, or sometimes support motion. Examples of base excitation, or where base excitation is considered, include:

- machines on rubber mounts
- automobiles excited by the road
• building under earthquake loading
• hospital equipment

Consider the following system base excited system

![Damped 1-DOF spring-mass system subjected to a displacement controlled base excitation showing the FBDs for the equilibrium and displaced positions.](image)

where $x$ is the displacement of the mass and $y$ is the displacement of the base. Note that we consider positive upward here. The EOM can be constructed the same as before, but now considering that the displacement of the springs and damper is $x - y$. In the equilibrium state, where a positive $x$ is up and the base displaces down:

$$\sum F_x = -k\delta - mg = 0 \quad (100)$$

Note that these are both negative because the base displacing down “pulls” the mass down with a force $k\delta$ (i.e. if you hold the mass and let the base “fall”). Conversely, the equation for the displaced state is:

$$\sum F_x = -k(\delta + x - y) - mg - c(\ddot{x} - \dot{y}) \quad (101)$$

Apply Newton’s second law about the mass ($m\ddot{x}$) of motion to the sum of forces for the displaced position we get:

$$\sum F_x = m\ddot{x} = -k\delta - kx + ky - mg - c\dot{x} + c\dot{y} \quad (102)$$

applying the equation $-k\delta - mg = 0$, and rearrange into the EOM yields:

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky \quad (103)$$
Now as before we assume an input for the base excitation. For simplicity we assume:

\[ y(t) = Y \sin(\omega_b t) \]  

(104)

Taking the derivative of the assumed input yields:

\[ \dot{y}(t) = Y \omega_b \cos(\omega_b t) \]  

(105)

where \( Y \) is the amplitude and \( \omega_b \) is the frequency of the base excitation. Adding these terms into our EOM yields:

\[ m\ddot{x} + c\dot{x} + kx = cY \omega_b \cos(\omega_b t) + kY \sin(\omega_b t) \]  

(106)

We can get this in standard form if we divide by \( m \) and apply the equations for the critical damping ratio and natural frequency:

\[ \ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = 2\zeta \omega_n \omega_b Y \cos(\omega_b t) + \omega_n^2 Y \sin(\omega_b t) \]  

(107)

This equation can be related to a spring-mass-damper system with two harmonic inputs, one cos and one sin as shown below:

\[ \ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = C \cos(\omega_b t) + D \sin(\omega_b t) \]  

(108)

where \( C \) and \( D \) are arbitrary coefficients.

### 3.5.1 Displacement Transmissibility Solution for Base Excitation

The steady-state solution is often of more important than the transient solution when designing systems for continuous use. The particular solution for the base excited system annotated in figure 15 with the EOM presented in equation 108 can be expressed as \( x_p(t) \). To solve for this expression we will use the linearity of the system and solve for a solution that is the sum of two particular solutions. Resulting in:

\[ x_p(t) = x_p^{(1)}(t) + x_p^{(2)}(t) \]  

(109)

Recall that the steady state solution for a harmonically excited spring-mass-damper can be expressed as \( x_p(t) = X \cos(\omega_b t - \phi) \). Therefore, for a base excited problem, the forcing function can be expressed as the sum of particular solutions:

\[ C \cos(\omega_b t) + D \sin(\omega_b t) = x_p = x_p^{(1)} + x_p^{(2)} \]  

(110)
where we dropped the \((t)\) term from the expression for simplicity in writing and:

\[
x_p^{(1)} = X^{(1)} \cos(\omega_b t - \phi_1)
\]
\[
x_p^{(2)} = X^{(2)} \sin(\omega_b t - \phi_1)
\]

Note that \(x_p^{(1)}\) uses a cos term while \(x_p^{(1)}\) uses a sin term. Both solutions use \(\phi_1\) as the damping term as the phase angle is independent of the excitation amplitude and the sin and cos terms account for the difference in phase. Now, for \(x_p^{(1)}\), we use the method of undetermined coefficients to obtain a solution for \(x_p^{(1)} = X^{(1)} \cos(\omega_b t - \phi_1)\). This can be as simple as setting \(2\zeta\omega_n\omega_b Y\) equal to \(f_0\) from equation 71 that defines \(X\) for underdamped systems. Again, \(2\zeta\omega_n\omega_b Y\) comes from the EOM in standard form as presented in equation 107. We can do this because both terms can be considered a “driving force”. This results in the equation:

\[
x_p^{(1)} = \frac{2\zeta\omega_n\omega_b Y}{\sqrt{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}} \cos(\omega_b t - \phi_1)
\]

where:

\[
\phi_1 = \tan^{-1}\left(\frac{2\zeta\omega_n\omega_b}{\omega_n^2 - \omega_b^2}\right)
\]

Next, the particular solution associated with \(x_p^{(2)} = X^{(2)} \sin(\omega_b t - \phi_1)\) can be obtained using the same method of undetermined coefficients an setting \(f_0\) from equation 71 to the driving force for \(x_p^{(2)}\) in equation 107, \(\omega_n^2\). This results in:

\[
x_p^{(2)} = \frac{\omega_n^2 Y}{\sqrt{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}} \sin(\omega_b t - \phi_1)
\]

As both equation 113 and 115 have the same argument \((\omega_b t - \phi_1)\), these can be added as:

\[
x_p = x_p^{(1)} + x_p^{(2)}
\]

to obtain:

\[
x_p = \omega_n Y \sqrt{\frac{\omega_n^2 + (2\zeta\omega_b)^2}{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}} \cos(\omega_b t - \phi_1 - \phi_2)
\]

and:

\[
\phi_2 = \tan^{-1}\left(\frac{\omega_n}{2\zeta\omega_b}\right)
\]
where $\phi_2$ is added to account for the cos and sin terms being combined. Again, the $(t)$ has been dropped for simplicity. As before, if we want to investigate how a frequency input will affect the response (frequency response) we can substitute

$$r = \frac{\omega_b}{\omega_n}$$  \hspace{1cm} (119)$$

into the temporal response to obtain:

$$X = Y \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}}$$  \hspace{1cm} (120)$$

Next, if we divide by $Y$ we can obtain a normalized expression for the displacement:

$$\frac{X}{Y} = \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}}$$  \hspace{1cm} (121)$$

Plotting this for several critical damping ratios:

![Figure 16: Displacement transmissibility for an underdamped 1-DOF system.](image)

Around resonance, the maximum amount of displacement is transmitted to the mass. Additionally, the above plot shows that at $r = \sqrt{2}$ the displacement trans-
missibility $X/Y$ is 1. Note the “flip” where overdamped systems have a greater response to excitations after $r = \sqrt{2}$ than do underdamped systems.

**Example 3.7** A very common example of base motion is the SDOF model of a vehicle wheel driving over a “rough” road as shown below.

![SDOF Model Diagram](image)

where $k = 400,000$ N, $m = 1007$ kg, $c = 20,000$ kg/s, the period of road roughness = 6m, and height of road roughness = 0.02 m. What is the deflection experience by the car at $v = 20$ km/h?

**Solution:** The road is applying a base excitation that can be approximated as

$$ Y = 0.01 \text{ m} \quad (122) $$

$$ v \text{ m/s} = 20 \text{ km/hr} \left( \frac{1000 \text{ m}}{1 \text{ km}} \right) \left( \frac{1 \text{ hours}}{3600 \text{ s}} \right) = \frac{50}{9} \text{ m/s} = 5.555 \text{ m/sec} \quad (123) $$

$$ \omega_b = \left( \frac{5.55 \text{ m}}{s} \right) \left( \frac{1 \text{ cycle}}{6 \text{ m}} \right) \left( \frac{2\pi \text{ rad}}{\text{cycle}} \right) = \frac{11.11\pi}{6} \text{ rad/s} = 5.817 \text{ rad/s} \quad (124) $$

Therefore, the sinusoidal for the base excitation is then:

$$ y(t) = (0.01)\sin(5.817t) \quad (125) $$
Next, we can calculate the natural frequency:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{400,000}{1007}} = 19.93 \text{ rad/s} \quad (126)$$

Therefore:

$$r = \frac{\omega_b}{\omega_n} = \frac{5.817}{19.93} = 0.292 \quad (127)$$

and:

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{20,000}{2\sqrt{1007 \cdot 400,000}} = 0.498 \quad (128)$$

Then it can be found that the maximum deflection of the car is: into the temporal response to obtain:

$$X = Y \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}} = Y \sqrt{\frac{1 + (2 \cdot 0.498 \cdot 0.292)^2}{(1 - 0.292^2)^2 + (2 \cdot 0.498 \cdot 0.292)^2}}$$

$$= 0.0108 \text{ m} \quad (129)$$

### 3.5.2 Force Transmissibility Solution for Base Excitation

For some systems, such as those with weak connections, the force transmitted to the mass is more important than the displacement of the mass. The force transmitted to the mass are the sums of the forces applied by the spring and damper. From the FBD above,

$$F(t) = k(x - y) + c(\dot{x} - \dot{y}) \quad (130)$$

where this force is counteracted by the inertial force of the mass:

$$F(t) = -m\ddot{x}(t) \quad (131)$$

Only considering the steady state we found that

$$x_p(t) = \omega_n Y \sqrt{\frac{\omega_n^2 + (2\zeta \omega_b)^2}{(\omega_n^2 - \omega_b^2)^2 + (2\zeta \omega_n \omega_b)^2}} \cos(\omega_b t - \phi_1 - \phi_2) \quad (132)$$

if we differentiate this twice, to obtain $\ddot{x}(t)$ and combine this with $F(t) = -m\ddot{x}(t)$ we get:

$$F(t) = m\omega_b^2 \omega_n Y \sqrt{\frac{\omega_n^2 + (2\zeta \omega_b)^2}{(\omega_n^2 - \omega_b^2)^2 + (2\zeta \omega_n \omega_b)^2}} \cos(\omega_b t - \phi_1 - \phi_2) \quad (133)$$
where the negative sign $F(t) = -m\ddot{x}(t)$ as the force transmitted to the mass is both positive and negative and we are solving for the amplitude of the transmitted force. Again applying:

\[ r = \frac{\omega_b}{\omega_n} \]  

(134)

this becomes:

\[ F(t) = F_T \cos(\omega_b t - \phi_1 - \phi_2) \]  

(135)

where $F_T$ is the magnitude of the transmitted force and is

\[ F_T = kY r^2 \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}} \]  

(136)

Again, this can be converted to a force transmissibility to provide a normalized response such that:

\[ \frac{F_T}{kY} = r^2 \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}} \]  

(137)

Plotting this for several critical damping ratios:

![Figure 17: Force transmissibility for an underdamped 1-DOF system.](image)

Figure 17: Force transmissibility for an underdamped 1-DOF system.
Again, note the key location $r = \sqrt{2}$. At $r = \sqrt{2}$ the force transmitted to the system is $2 \frac{F_T}{k_Y}$. However, also note that the normalized force does not necessarily fall off for $r$ values greater than $r = \sqrt{2}$.

**Example 3.8** For the system given below and excited at the base, should the system be excited above or below the natural frequency if the transmitted force is the design limitation? Consider the under damped with $\zeta = 0.1$, and the over damped with $\zeta = 2$ conditions.

![Diagram of a 1-DOF system](image)

**Figure 18:** Force transmissibility for an underdamped 1-DOF system.

**Solution:** We can plot the transmissibility of both the force and displacement onto one plot. For $\zeta = 0.1$
Figure 19: Force and displacement transmissibility for the considered base excited system with $\zeta = 0.1$.

It is clear that to minimize the force, the system should be driven with a frequency below the natural frequency. Next for $\zeta = 2$:

Figure 20: Force and displacement transmissibility for the considered base excited system with $\zeta = 2$.

It can be seen that the same rationale applies. Therefore, for both $\zeta = 0.1$ and $\zeta = 2$ the system should be excited below the natural frequency.
Example 3.9 A single story building is subjected to a harmonic ground motion, $\ddot{y}(t) = A\cos(\omega_b t)$. a) Find the steady-state solution for the structure. b) If a damper was added between the base and the floor, and $r = 2$, what would be the ideal critical damping coefficient to insure the safety of the building. (Think of safety as limiting displacement and transmitted force.)

Solution (a):
For simplicity, we can rearrange the system as what follows:

\[
\begin{align*}
m\dddot{x} + kx &= ky \\
\end{align*}
\]

(solving for the EOM yields:

\[
m\ddot{x} + kx = ky
\] (138)

Notice that this is the same as the EOM for a damped 1-DOF system if $c = 0$.

\[
m\ddot{x} + c\dot{x} + kx = +c\dot{y} + ky \rightarrow m\ddot{x} + kx = ky
\] (139)

Therefore, we can use the solution:

\[
x_p(t) = \omega_n Y \sqrt{\frac{\omega_n^2 + (2\zeta \omega_b)^2}{(\omega_n^2 - \omega_b^2)^2 + (2\zeta \omega_n \omega_b)^2}} \cos(\omega_b t - \phi_1 - \phi_2)
\] (140)
where:
\[
\phi_1 = \tan^{-1}\left(\frac{2\zeta \omega_n \omega_b}{\omega_n^2 - \omega_b^2}\right)
\]
\[
\phi_2 = \tan^{-1}\left(\frac{\omega_n}{2\zeta \omega_b}\right)
\]

Now we have, or can easily get, values for \(\omega_n\), \(\omega_b\), and \(\zeta\). However, we do not have an expression for \(Y\). We can extract the displacement (and therefore the \(Y\)) from the acceleration as:
\[
\ddot{y}(t) = A \cos(\omega t)
\]
\[
\dot{y}(t) = \frac{A}{\omega} \sin(\omega t) + C_1
\]
\[
y(t) = -\frac{A}{\omega^2} \cos(\omega t) + C_1 t + C_2
\]

Resulting in
\[
Y = -\frac{A}{\omega^2}
\]

**Solution (b):** From the plots we solved for before, we can see that we want a critical damping coefficient that is as low as possible. This means any damping added to the system will decrease its safety. This may seem counter-intuitive, but this is because we are attempting to drive the structure at a frequency higher than its natural frequency, something that does not commonly happen. Typically excitations for a structure are well below its natural frequency.
4 Transfer Function for Vibrating Systems

Thusfar, this text has only considered forced vibrations for 1-DOF systems excited with forcing functions that can be easily expressed using either sin or cos examples. Therefore, the previously developed solutions are only acceptable for systems with known and simple excitations. This chapter will introduce the concept of transfer functions for solving vibration related problems. The transfer function, in particular the Laplace transfer function, is an important tool in the study of vibrations as it allows the practitioner to solve for the temporal response of a system for a variety of inputs using a single approach. Examples of force excitation that can be calculated include using this method include:

- sinusoidal
- base excitation
- impulse
- arbitrary input
- arbitrary periodic input

4.1 Transfer Function Method (Generic)

Consider the following system

\[ F(s) \rightarrow H(s) \rightarrow X(s) \]

where \( F(s) \) is the input, \( H(s) \) is the system, and \( X(s) \) is the output from the system. This formulation is called the transfer-function approach and is commonly used for the formulation and solution of dynamic problems in the control literature. It can also be used for solving the various forced-vibration problems including those from complex or stochastic inputs.

**Review 4.1** Laplace transforms, or more broadly integral transform, are a procedure for integrating the time dependence of a function into a function of an alternative variable, or parameter. These functions can often be manipulated algebraically.
Of interest to this class is the Laplace transform ($L[\ ]$) of the function $f(t)$, expressed as $L[f(t)]$. Here, a Laplace transform is used as a method of solving the differential equations of motion by reducing the computation needed to that of integration and algebraic manipulation.

The definition of the Laplace transform of the function $f(t)$ is:

$$L[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt \quad (1)$$

where $s$ represents a variable in the complex plane and $f(t) = 0$ for all values of $t < 0$. Here, the $s$ is a complex value. Lastly, the term $F(s)$ is a generic term that represents the input to a system. As this class needs the derivative of functions, we will calculate that next:

$$L[\dot{f}(t)] = \int_0^\infty \dot{f}(t)e^{-st}dt = \int_0^\infty e^{-st}\frac{df(t)}{dt}dt \quad (2)$$

Integration by parts yields

$$L[\dot{f}(t)] = e^{-st}f(t)\bigg|_0^\infty + s\int_0^\infty e^{-st}f(t)dt \quad (3)$$

Astutely, it can be noticed that the second term $s\int_0^\infty e^{-st}f(t)dt$ is the input to the system $F(s)$. Therefore, with a little rearranging this becomes:

$$L[\dot{f}(t)] = sF(s) - f(0) \quad (4)$$

Successive iteration yields:

$$L[\ddot{f}(t)] = s^2F(s) - sf(0) - \dot{f}(0) \quad (5)$$

A few key points of the Laplace transforms are:

- the domain of the problem from the real number line ($t$) to the complex plane ($s$).
- The integration of the Laplace transform changes differentiation into multiplication.
• The transform procedure is linear. Therefore, the transform of the linear combination of two transforms is the same as the linear transformation of these functions.

• to move from the time domain to the complex number plane we typically use tables of pre-solved integral.

• The function \( x(t) \) can be obtained by taking the inverse Laplace transform defined as \( x(t) = \mathcal{L}^{-1}[X(s)] \)

The Laplace transform can be calculated in symbolic form. In particular interest to this text is the Laplace form of the system input \( F(s) \) and output \( X(s) \). To expand the symbolic form of the Laplace transform for the system inputs are and for system outputs:

\[
\mathcal{L}[f(t)] = F(s) \tag{6}
\]

\[
\mathcal{L}[\dot{f}(t)] = sF(s) - x(0) \tag{7}
\]

\[
\mathcal{L}[\ddot{f}(t)] = s^2F(s) - sx(0) - \dot{x}(0) \tag{8}
\]

Here, \( f(0) \) and \( \dot{f}(0) \) are the initial values of the function \( f(t) \). And for system outputs:

\[
\mathcal{L}[x(t)] = X(s) \tag{9}
\]

\[
\mathcal{L}[\dot{x}(t)] = sX(s) - x(0) \tag{10}
\]

\[
\mathcal{L}[\ddot{x}(t)] = s^2X(s) - sx(0) - \dot{x}(0) \tag{11}
\]

Here, \( x(0) \) and \( \dot{x}(0) \) are the initial values of the function \( x(t) \).

### 4.2 Transfer Function Method for Solving Vibrating Systems

As mentioned in the introduction to this chapter, a variety of systems can be solved for using the transfer function method. The procedure for using the Laplace transform to solve equations of motion expressed as an inhomogeneous ordinary differential equation is:

1. Take the Laplace transform of both sides of the EOM while treating the time derivatives symbolically.

2. Solve for \( X(s) \) in the obtained equation.
3. Apply the inverse transform \[ x(t) = \mathcal{L}^{-1}[X(s)] \]

### 4.2.1 Free Vibration for Undamped Systems

Consider the undamped single-DOF system:

![Figure 1: A spring mass model of a 1-DOF system.](image)

The EOM for this system is a homogeneous differential equation because the right-hand side is equal to zero:

\[ m\ddot{x}(t) + kx(t) = 0 \]  

(12)

Here we will leave the “\( t \)” for clarity to differentiate the time domain solution from Laplace solution “\( s \)” in the complex plane, as discussed in review 4.1. The EOM can be rewritten in standard form as:

\[ \ddot{x}(t) + \omega_n^2 x(t) = 0 \]  

(13)

where the initial conditions at \( t = 0 \) are \( x(0) = x_0 \) and \( \dot{x}(0) = v_0 \). Taking the Laplace transforms, in symbolic form using equations 9 - 11, of both sides of the EOM yields:

\[ [s^2X(s) - sx_0 - v_0] + [\omega_n^2 X(s)] = 0 \]  

(14)

using equations 9 and 11 from section 4.1. Solving for the output of the system \( X(s) \) yields:

\[ X(s) = \frac{sx_0 + v_0}{s^2 + \omega_n^2} \]  

(15)

The full derivation of the above step is presented in appendix [X]. We can expand this form of \( X(s) \) to obtain equations listed in our Laplace Transform table:

\[ X(s) = \frac{sx_0}{s^2 + \omega_n^2} + \frac{v_0}{s^2 + \omega_n^2} \cdot \frac{\omega_n}{\omega_n} \]  

(16)

This becomes:

\[ X(s) = x_0 \frac{s}{s^2 + \omega_n^2} + \left( \frac{v_0}{\omega_n} \right) \cdot \frac{\omega_n}{s^2 + \omega_n^2} \]  

(17)
Next, using the inverse Laplace transform \( x(t) = \mathcal{L}^{-1}[X(s)] \) and the two following Laplace transforms (#5 and #6):

\[
\begin{align*}
    f(t) \text{ is } \cos(\omega t) \text{ when } F(s) &= \frac{s}{s^2 + \omega^2} \\
    f(t) \text{ is } \sin(\omega t) \text{ when } F(s) &= \frac{s^2}{s^2 + \omega^2}
\end{align*}
\]

(18) (19)

Therefore, we can obtain the solution for the system output \( X(s) \) as:

\[
x(t) = x_0 \cos(\omega_n t) + \frac{v_0}{\omega_n} \sin(\omega_n t)
\]

(20)

The same procedure works for calculating the under damped and forced responses, however, in these responses calculating the algebraic solution for \( X(s) \) often results in quotients of polynomials in \( s \). These Polynomial ratios may not be found in simple Laplace tables and must be solved using the method of partial fractions. An example of this procedure can be found in Appendix B of Inman.

### 4.2.2 Impulse Response Function

A very common source of vibration is the sudden application for a short-duration force called an impulse. An impulse excitation is a force that is applied for a very short, or infinitesimal, length of time and represents one example of a shock loading. An impulse is a nonperiodic force that is represented by the symbol \( \delta \).

The response of a system to an impulse is identical to the free response of the system to certain initial conditions. This is illustrated in the following, in many useful situations the applied force \( F(t) \) is impulsive in nature (i.e., acts with large magnitude for a very short period of time).

![Figure 2: An impulse function with the impulse at \( t = 0 \).](image-url)
The impulse response function can be solved for analytically, however, we will solve it using the transfer function approach. Here we will consider the under-damped spring-mass system. First, assume that the system is at rest (no initial conditions). Next, we write the EOM as:

\[ m\ddot{x} + c\dot{x} + kx = \delta(t) \]  

Taking the Laplace transform of both sides of the equation yields

\[ m(s^2X(s) - sx(0) - \dot{x}(0)) + c(sX(s) - x(0)) + kX(s) = 1 \]  

Note that the \( \mathcal{L}[\delta] = 1 \) per #1 in the transform table. However, if we assume zero initial conditions (a system at rest when the impulse happens), the equation simplifies to.

\[ ms^2X(s) + csX(s) + kX(s) = 1 \]  

or

\[ (ms^2 + cs + k)X(s) = 1 \]  

Solving this equation for \( X(s) \):

\[ X(s) = \frac{1}{m} \cdot \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]  

Again, the mass is extracted to develop a formulation that can be found in the Laplace tables. Setting the constraint that \( \zeta < 1 \) and consulting #10 in the table for Laplace transforms results in:

\[ x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t) \]  

where this is the general solution for a damped system subjected to an impulse loading function. For the undamped case a solution can be obtained by setting \( \zeta = 0 \). This results in the following form for the undamped case:

\[ x(t) = \frac{1}{m\omega_n} \sin(\omega_n t) \]  

Below is a typical response for both a undamped and underdamped 1-DOF system subject to an impulse response at \( t = 0 \) seconds.
4.2.3 Unit step function

Now consider a unit step function \( \Phi \):

A step function is a common loading situation and can represent the dropping of a load into a truck, a car going over a curve, or a motor starting up.

The Laplace transform of the function, for a unit step function \( \Phi \), is:

\[
\mathcal{L}[\Phi(t)] = \int_0^\infty e^{-st} dt = -\frac{e^{-\infty}}{s} + \frac{e^{0}}{s} = \frac{1}{s}
\]
This also lines up with Laplace Transform # 3 from the attached table. This would be expected as $\Phi$ is used to represent the unit step function (i.e. a step function with a displacement of 1). As we consider linear systems in this class, we can scale the magnitude of the response by the magnitude of the impulse after the transform is performed.

### 4.2.4 Undamped spring-mass system

For a spring-mass system subjected to a unit step, assuming both initial conditions are zero, the solution can be obtained using the transform method. First, the EOM is

$$m\ddot{x}(t) + kx(t) = \Phi(t) \quad (28)$$

Taking the Laplace transform of both sides and assuming zero initial conditions yields:

$$ms^2X(s) + kX(s) = \frac{1}{s} \quad (29)$$

Next, this equation is solved for $X(s)$ as:

$$X(s) = \frac{1}{s(ms^2 + k)} \quad (30)$$

This can be rearranged as:

$$X(s) = \frac{1}{m} \cdot \frac{1}{s(s^2 + \omega_n^2)} \quad (31)$$

where $\frac{1}{m}$ will pass through the Laplace function. Therefore, taking the inverse Laplace transform yields:

$$x(t) = \frac{1}{m} \cdot \frac{1}{\omega_n} (1 - \cos(\omega_n t)) = \frac{1}{k} (1 - \cos(\omega_n t)) \quad (32)$$

### 4.2.5 Under damped spring-mass system

For a spring-mass-damper system subjected to a unit step, assuming both initial conditions are zero, the solution can be obtained using the transform method. First, the EOM is:

$$m\ddot{x}(t) + c\dot{x} + kx(t) = \Phi(t) \quad (33)$$
Converting to the standard form results in:

\[ \ddot{x}(t) + 2\zeta \omega_n \dot{x} + \omega_n^2 x(t) = \frac{1}{m} \cdot \Phi(t) \quad (34) \]

taking the Laplace transform of both sides and assuming zero initial conditions yields:

\[ s^2 X(s) + 2\zeta \omega_n s X(s) + \omega_n^2 X(s) = \frac{1}{m} \cdot \frac{1}{s} \quad (35) \]

Next, this equation is solved for \( X(s) \) as:

\[ X(s) = \frac{1}{s^2 + 2\zeta \omega_n s + \omega_n^2} \cdot \frac{1}{m} \cdot \frac{1}{s} \quad (36) \]

multiplying the right-hand-side of this equation by \( \frac{\omega_n^2}{\omega_n^2} \) results in:

\[ X(s) = \frac{1}{m \omega_n^2} \cdot \frac{\omega_n^2}{s(s^2 + 2\zeta \omega_n s + \omega_n^2)} \quad (37) \]

Again, the \( \frac{1}{m \omega_n} \) will pass through the Laplace function. Therefore, taking the inverse Laplace transform yields:

\[ x(t) = \frac{1}{m \omega_n^2} \cdot \left( 1 - \frac{\omega_n}{\omega_d} e^{-\zeta \omega_n t} \sin(\omega_d t + \phi) \right) \]

where \( \phi = \cos^{-1}(\zeta) \), where \( \zeta < 1 \) \quad (38)

After obtaining equations for the undamped and under damped cases, the responses for the unit step, solved with the transform method, can be plotted as:

Figure 5: Temporal responses from a underdamped and undamped 1-DOF systems to a impulse response function.
Note that the system will settle out around $F_0/k$ where $F_0 \Phi$ is a scaling factor for the step loading.

**Example 4.1** A load of roosters $m_r$ is dropped on the floor of a truck bed. Assuming the roosters do not move and that the truck bed is modeled as a spring-mass-damper system (of values $k$, $m$, and $c$, respectively). The load is modeled as a force $F(t) = m_r g$ applied to the spring-mass-damper system, as illustrated in the following figure. This allows a crude analysis of the response of the truck’s suspension. First assume that the truck’s damper is broken, how does the maximum dynamic displacement compare to the static displacement. What would happen to the maximum displacement if the damper was repaired on the truck?

![Figure 6: Dump truck being loaded with roosters showing (a) roosters going into the truck bed; and (b) the single-degree-of-freedom vibration model.](image)

**Solution:** First, setting the load applied to the truck as 1 unit, it can be seen that this is a unit step loading condition and a broken damper represents an undamped case. To obtain a rough idea about the nature of static and dynamic displacement, the undamped displacement is

$$dx(t) = \frac{1}{k} \left(1 - \cos(\omega_n t)\right) = \frac{m_r g}{k} \left(1 - \cos(\omega_n t)\right) \quad (39)$$

This equation has a maximum amplitude when the $\cos(\omega t) = -1$, resulting in:

$$x(t) = \frac{m_r g}{k} (1 - (-1)) \quad (40)$$

This can be rearranged for the maximum displacement value $x_{max}$ as:

$$x_{max} = 2 \frac{m_r g}{k} \quad (41)$$
This is twice the static displacement (i.e., twice the distance the truck would be deflected if the roosters were placed gently and slowly onto it). Thus, if the truck were designed with springs based only on the static load, with no margins of safety, the springs in the truck would potentially break, or permanently deform, when subjected to the same mass applied dynamically (e.g., dropped) into the truck. Hence, it is important to consider the vibration (dynamic) response in designing structures that could experience dynamic loading.

4.2.6 Arbitrary Inputs to a System

The time domain response of a system to an arbitrary input force in time can be calculated using a series of impulses as shown in figure 7. This method allows the practitioner to easily calculate the response of an arbitrary input to a system using a single expression executed in a ”for loop”. This type of analysis is often more efficient in terms of programing than more direct methods such as the transfer functions sown in this text.

Figure 7: Generalized response showing that any signal can be represented as a series of impulse signals.

**Example 4.2** In testing, an hammer is used to excite a 1-DOF system with an impact (i.e. impulse), however, the hammer ascendantly impacts the system twice. The first impact has a force of 0.2 N, while the second has a force of 0.1 N and happens 0.1 seconds after the first impact. Plot the response for the double impact. The system has the parameters $m = 1$ kg, $c = 0.5$ kg/s, $k = 4$ N/m.

**Solution:** First, we can define the forcing function as:

$$F(t) = 0.2\delta(t) + 0.1\delta(t - \tau)$$ (42)
where $\tau$ is the offset between the first and second impacts. Next, considering that the unit impulse has a magnitude of 1 we can obtain solutions for the first impact by first writing it’s EOM:

$$m\ddot{x} + c\dot{x} + kx = 0.2\delta(t)$$  \hspace{1cm} (43)

Taking the Laplace transform of both sides of the equation yields

$$m\left(s^2X(s) - sx(0) - \dot{x}(0)\right) + c\left(sX(s) - x(0)\right) + kX(s) = 0.2$$  \hspace{1cm} (44)

However, assuming zero initial conditions, the equation simplifies to.

$$(ms^2 + cs + k)X(s) = 0.2$$  \hspace{1cm} (45)

Solving this equation for $X(s)$:

$$X(s) = \frac{0.2}{m} \cdot \frac{1}{s^2 + 2\zeta \omega_n s + \omega_n^2}$$  \hspace{1cm} (46)

Again, consulting #10 in the table for Laplace transforms results in:

$$x_1(t) = \frac{0.2}{m\omega_d} e^{-\zeta \omega_n t} \sin(\omega_d t)$$  \hspace{1cm} (47)

where this is the general solution for a damped system subjected to an impulse loading function. The second impact can now be solved for using the same method. However, now the time ($t$) must be offset by ($\tau$) to allow the impact to still be located at $t = 0$ in terms of the second impact. This results in:

$$x_1(t) = \frac{0.2}{m\omega_d} e^{-\zeta \omega_n (t-\tau)} \sin(\omega_d (t - \tau))$$  \hspace{1cm} (48)

$$x_2(t) = \frac{0.1}{m\omega_d} e^{-\zeta \omega_n (t-\tau)} \sin(\omega_d (t - \tau))$$  \hspace{1cm} (49)

Next, using the knowledge that the systems are linear and that the Laplace transform of a linear combination of two transforms is the same as the linear transformation of these functions we can build the piecewise function:
\[ x(t) = \begin{cases} 
0.2 \frac{m}{\omega_d} e^{-\zeta \omega_d t} \sin(\omega_d t) & \text{if } t < \tau \\
0.2 \frac{m}{\omega_d} e^{-\zeta \omega_d t} \sin(\omega_d t) + 0.1 \frac{1}{m \omega_d} e^{-\zeta \omega_d (t-\tau)} \sin(\omega_d (t-\tau)) & \text{if } \tau \leq t
\end{cases} \]

For the mass, damping, and stiffness values given above this can be plotted as:

### 4.3 Transform function for response to random inputs

#### 4.3.1 Defining the transfer function \( H(s) \)

Again, consider the following system

For this system representation, \( F(s) \) is the Laplace of the transform of the driving force and \( H(s) \) is the Laplace transform of the response of the system \( h(t) \).

We need to define transfer function \( H(s) \) for a generic system. To do this let us show the reasoning behind the transfer function. Here we will show that the output of any system \( (x(t)) \) can be related to the input of the system \( (f(t)) \) through a series of polynomial coefficients \( (a \text{ and } b) \). Consider the general \( n^{th} \)-order linear, time-invariant differential equation that governs the behavior of the dynamic system.

\[ a_n \frac{d^n x(t)}{dt^n} + a_{n-1} \frac{d^{n-1} x(t)}{dt^{n-1}} + \ldots + a_0 x(t) = b_m \frac{d^m f(t)}{dt^m} + b_{m-1} \frac{d^{m-1} f(t)}{dt^{m-1}} + \ldots + b_0 f(t) \]  

(50)
where \( x(t) \) is the output and \( f(t) \) is the input. Note that this is similar to the formulation we have had before for the EOM. Taking the Laplace transformation of both side of the above equation yields

\[
a_n s^n X(s) + a_{n-1} s^{n-1} X(s) + \ldots + a_0 X(s) + \text{initial condition for } x(t) = 0 \tag{51}
\]

\[
b_m s^m F(s) + b_{m-1} s^{m-1} F(s) + \ldots + b_0 F(s) + \text{initial condition for } f(t) = 0
\]

It can be seen that this equation is a purely algebraic expression. If we assume the initial conditions to be zero, the equation reduces to the following:

\[
(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0) X(s) = (b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0) F(s) \tag{52}
\]

if we rearrange equation 52 to solve for the relationship between the Laplace variables \( X(s) \) and \( F(s) \) and the algebraic expressions we get:

\[
\frac{X(s)}{F(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0} \tag{53}
\]

this shows that the ratio of the input algebraic expressions over the output algebraic expressions is equal to the ratio of the output Laplace variable over the input Laplace variable. This shows that we can relate the Laplace variables to the algebraic expressions. Therefore, we can define the transfer function \( H(s) \) as:

\[
H(s) = \frac{X(s)}{F(s)} \tag{54}
\]

In a more formal term, the transfer function that is defined as: “The ratio of the Laplace transforms of the output or response function to the laplace transform of the input or forcing function assuming zero initial conditions”.

Equation 54 can be rearranged to show that the output of the system \( X(s) \), can be obtained if we know the input \( F(s) \) and the transfer function \( H(s) \):

\[
X(s) = H(s) F(s) \tag{55}
\]

4.3.2 Transfer Function method (Steady-State solution)

Considering the forced system:
that can be expressed as the equation of motion

\[ m\ddot{x} + c\dot{x} + kx = F_0 \cos(\omega t) \] (56)

Here \( F_0 \cos(\omega t) \), is used at the input but any input will develop the same transfer function as the transfer function is bounded to the system and not the input. From the #6 in the table for Laplace Transforms, we know that

\[ \mathcal{L}[\cos(\omega t)] = \frac{s}{s^2 + \omega^2} \] (57)

Therefore,

\[ F(s) = \frac{F_0 s}{s^2 + \omega^2} \] (58)

Ignoring the initial conditions, and therefore considering only the particular solution, and taking the Laplace transform of the EOM equation yields:

\[ (ms^2 + cs + k)X(s) = \frac{F_0 s}{s^2 + \omega^2} \] (59)

where \( s \) is the complex transform variable and \( X(s) \) denotes the Laplace transform of the unknown function \( x(t) \). Solving algebraically for the \( X(s) \) yields:

\[ X(s) = \frac{F_0 s}{(ms^2 + cs + k)(s^2 + \omega^2)} \] (60)

Now that we have \( F(s) \) and \( X(s) \) we can obtain \( H(s) \) as

\[ H(s) = \frac{X(s)}{F(s)} = \frac{F_0 s}{(ms^2 + cs + k)(s^2 + \omega^2)} \cdot \frac{s^2 + \omega^2}{F_0 s} = \frac{1}{ms^2 + cs + k} \] (61)

or

\[ H(s) = \frac{1}{ms^2 + cs + k} \] (62)

This ratio is termed the transfer function of a system and is an important tool in vibration analysis.
Recall that $s$ is a complex number, if the value of $s$ is restricted to lie along the imaginary axis in the complex plane (i.e., if $s = j\omega$), the transfer function becomes:

$$H(j\omega) = \frac{1}{m(j\omega)^2 + cj\omega + k} = \frac{1}{-m\omega^2 + cj\omega + k}$$

(63)

rearranging into a standard form yields:

$$H(j\omega) = \frac{1}{k - m\omega^2 + c\omega j}$$

(64)

recall that $j^2 = -1$. This is the frequency response function of the system. Therefore, it can be seen that the frequency response function of the system is the transfer function of the system evaluated along the imaginary axis $s = j\omega$. However, this expression contains imaginary values (that help to account for the phase in the system) and therefore can be challenging to work with. As the amplitude $|H(j\omega)|$ of the response (the real portion of the equation) is useful to the practitioner, it is prudent to consider the special case of amplitude response while neglecting the phase response. Consider that:

$$H(j\omega) = R + Ij$$

(65)

so

$$|H(j\omega)| = \sqrt{R^2 + I^2}$$

(66)

multiplying $H(j\omega)$ by its complex conjugate yields:

$$H(j\omega) = \left(\frac{1}{k - m\omega^2 + c\omega j}\right) \left(\frac{k - m\omega^2 - c\omega j}{k - m\omega^2 - c\omega j}\right)
= \left(\frac{k - m\omega^2}{(k - m\omega^2)^2(c\omega)^2}\right) \left(\frac{-c\omega}{(k - m\omega^2)^2(c\omega)^2 j}\right)$$

(67)

therefore, $R = \frac{k - m\omega^2}{(k - m\omega^2)^2(c\omega)^2}$ and $I = \frac{-c\omega}{(k - m\omega^2)^2(c\omega)^2}$. Now, calculating the amplitude of $H(j\omega)$ we get:

$$H(\omega) = |H(j\omega)| = \sqrt{R^2 + I^2} = \sqrt{\frac{(k - m\omega)^2 + (-c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2}}$$

$$= \sqrt{\frac{1}{(k - m\omega^2)^2 + c^2\omega^2}} = \frac{1}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}$$

(68)
where \( H(\omega) \) represents only the amplitude of the frequency response function and therefore drops the \( j \) term from the expression.

To recap, for a single DOF damped spring-mass system the transfer function is:

\[
H(s) = \frac{1}{ms^2 + cs + k} \tag{69}
\]

And the frequency response function is:

\[
H(j\omega) = \frac{1}{k - m\omega^2 + c\omega j} \tag{70}
\]

While the amplitude of the frequency response is:

\[
H(\omega) = |H(j\omega)| = \frac{1}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \tag{71}
\]

**Example 4.3** Considering the forced system:

![Spring-dashpot-mass model](image)

Figure 9: A spring-dashpot-mass model of a 1-DOF system with external excitation.

Set the forcing function to be \( F_0 \sin(\omega t) \) and calculate the transfer function.

**Solution:** The equation of motion for the system is:

\[
m\ddot{x} + c\dot{x} + kx = F_0 \sin(\omega t) \tag{72}
\]

From the #6 in the table for Laplace Transforms, we know that:

\[
\mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2 + \omega^2} \tag{73}
\]

Therefore,

\[
F(s) = \frac{F_0 \omega}{s^2 + \omega^2} \tag{74}
\]
Ignoring the initial conditions and taking the Laplace transform of the EOM equation yields:

\[(ms^2 + cs + k)X(s) = \frac{F_0 \omega}{s^2 + \omega^2}\]  

(75)

Solving algebraically for the \(X(s)\) yields:

\[X(s) = \frac{F_0 \omega}{(ms^2 + cs + k)(s^2 + \omega^2)}\]  

(76)

Now that we have \(F(s)\) and \(X(s)\) we can obtain \(H(s)\) as

\[H(s) = \frac{X(s)}{F(s)} = \frac{F_0 \omega}{(ms^2 + cs + k)(s^2 + \omega^2)} \cdot \frac{s^2 + \omega^2}{F_0 \omega} = \frac{1}{ms^2 + cs + k}\]  

(77)

or

\[H(s) = \frac{1}{ms^2 + cs + k}\]  

(78)

This is identical to the solution obtained using \(F_0 \cos(\omega t)\) as would be expected because the transfer function is related to the system and not to the input.

### 4.3.3 Response to Random Inputs

The transfer and frequency response functions can be very useful for determining the system’s response to random inputs. Up to this point we have solved for deterministic input.

- **Deterministic**- For a known time \(t\), the value of the input force \(F(t)\) is precisely known.

- **Random** For a known time \(t\), the value of the input force \(F(t)\) is known only statistically.

To expand, a random signal is a signal with no obvious pattern. For these types of it is not possible to focus on the details of the input signal, as is done with a deterministic signal, rather the signal is classified and manipulated in terms of its statistical properties.

Randomness in vibration analysis can be thought of as the result of a series of results obtained from testing a system repeatability for various inputs under varying conditions. In these cases, one record or time history is not enough to
describe the system. Rather, an ensemble of various tests are used to describe how the system will respond to the various inputs.

First, let us consider two inputs, a deterministic input (typical sin wave), and a random input (white noise). These inputs are shown in figure 10.

![Figure 10: Two arbitrary inputs: (a) a sinusoidal; and (b) uniform random noise.](image)

One of the first factors to consider is the mean of the random signal \( x(t) \), defined as:

\[
E[x] = \bar{x} = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t) dt
\]

where \( T \) is the length in time of the data collected. However, for random signals we often want to consider signals with an average mean of zero (i.e. \( \bar{x}(t) = 0 \)). Therefore, for signals not centered around zero we can obtain a zero centered signal if the signal is stationary and we subtract the mean value from \( \bar{x} \) from the signal \( x(t) \). This can be written as:

\[
x'(t) = x(t) - \bar{x}
\]

where the \( x'(t) \) is now centered around zero. As mentioned before, it is important to consider whether or not the input signals are stationary. A signal is stationary if its statistical properties (usually expressed by its mean) do not change with time. Here, it can be seen that for our inputs considered the signals are stationary if a long enough time period is considered.
Another important variable is variance (or mean-square value) of the random variable \( x(t) \) defined as:

\[
E[(x - \bar{x})^2] = \lim_{T \to \infty} \frac{1}{T} \int_0^T (x(t) - \bar{x})^2 dt
\]

and provides a measure of the magnitude of the fluctuations in the signal \( x(t) \). If the signal has an expected value of zero, or \( E[x] = 0 \), this simplifies to.

\[
E[x^2] = \bar{x}^2 = \lim_{T \to \infty} \frac{1}{T} \int_0^T x^2(t) dt
\]

This expression leads to an to calculate the root-mean-square (RMS) of the signal as:

\[
x_{\text{rms}} = \sqrt{\bar{x}^2}
\]

Considering a nonstationary signal, a important measure of interests is how fast the value of the variables change. This is important to understand as it address how long it takes to measure enough samples of the variable before a meaningful statistical value can be calculated. One way to quantify how fast the values of signal change is the autocorrelation function:

\[
R_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)x(t + \tau) dt
\]

The subscript \( xx \) denotes that this is a measure of the response for the variable \( xx \), \( \tau \) is the time difference between the values at which the signal \( x(t) \) is sampled. The auto collation for the two inputs considered above are expressed in figure 11.
Figure 11: Responses from the autocorrelation function for the inputs shown in figure 10 showing: (a) a sinusoidal; and (b) uniform random noise.

Note that the value of $\tau$ selected in the autocorrelation function greatly affects its response for the sinusoidal input. This is because the values for the sinusoidal are highly correlated. To expand, the value at any time $t$ is greatly affected by the values immediately before and after it. This is not the case for the random input where the signal is not correlated and therefore there is little difference in changing the value of $\tau$ on the response of the autocorrelation function.

Next, if we take the Fourier transform of the autocorrelation function we obtain the power spectral density (PSD) defined as:

$$S_{xx}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau$$  \hspace{1cm} (85)$$

where the integral of $R_{xx}(\tau)$ changes the real number $\tau$ into the frequency-domain value $\omega$. The frequency spectrum (hence the $S$ in $S_{xx}(\omega)$) for the two input cases considered are plotted in figure 12.
Figure 12: Power spectral density plots for the inputs shown in figure 10 showing: (a) a sinusoidal; and (b) uniform random noise.

where the flat frequency response for the random input denotes that the random input is white noise input. This flat frequency response in the frequency domain can be denoted $S_0$, such that $S_{ff}(\omega) = S_0$ or $S_{xx}(\omega) = S_0$, depending on whether the frequency spectrum of the input ($ff$) or output ($xx$) is being considered. While a true white noise input would be perfectly flat, white noise is really just a theoretical concept as all real-world data will have some variation in the frequency domain as diagrammed in figure 12(b).

Recall that $S_{xx}$ is the spectrum of the response of the system. For the one-DOF system considered here, we can express the arbitrary input as a series of impulse inputs as discussed in section 4.2.6. This knowledge, along with the frequency response function can be used to relate the spectrum of the input $S_{ff}(\omega)$ to the output through the transfer function as:

$$S_{xx}(\omega) = |H(j\omega)|^2 \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{ff}(\tau) e^{-j\omega \tau} d\tau \right]$$

This can also be expressed in symbolic form as:

$$S_{xx}(\omega) = |H(j\omega)|^2 S_{ff}(\omega)$$

where $R_{ff}$ denotes the autocorrelation function of $F(t)$ and $S_{ff}$ denotes the PSD of the forcing function $F(t)$. The notation $|H(j\omega)|^2$ is the square of the magnitude of the complex frequency response function. A more detailed derivation can be
found in [Engineering Vibrations, Inman (2001)], [Random Vibrations, Spectral & Wavelet Analysis, Newland (1993)], but here it is more important to study the results rather than the derivations.

**Example 4.4** Consider the following system

![Figure 13: A spring-dashpot-mass model of a 1-DOF system with external excitation.](image)

Calculate the PSD of the response \( x(t) \) given that the PSD of the applied force \( S_{ff}(\omega) \) is white noise.

**Solution:** From the system we know that the EOM is

\[
m\ddot{x} + c\dot{x} + kx = F(t)
\]  \hspace{1cm} (88)

The frequency response function for this system is

\[
H(j\omega) = \frac{1}{k - m\omega^2 + c\omega j}
\]  \hspace{1cm} (89)

while the amplitude of the response is:

\[
H(\omega) = |H(j\omega)| = \frac{1}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}
\]  \hspace{1cm} (90)

Applying the equation that relates \( S_{ff}(\omega) \) to \( S_{xx}(\omega) \) we get:

\[
S_{xx}(\omega) = |H(j\omega)|^2S_{ff}(\omega) = \left| \frac{1}{k - m\omega^2 + c\omega j} \right|^2 S_{ff}(\omega)
\]  \hspace{1cm} (91)

White noise means the forcing function \( S_{ff}(\omega) \) is constant across the frequency spectrum, therefore, \( S_{ff}(\omega) = S_0 \). Additionally as:

\[
|H(j\omega)|^2 = \left| \frac{1}{k - m\omega^2 + c\omega j} \right|^2 = \frac{1}{(k - m\omega^2)^2 + c^2\omega^2}
\]  \hspace{1cm} (92)
where the absolute value is the amplitude of the system. Therefore, we obtain:

\[
S_{xx}(\omega) = |H(j\omega)|^2 S_0 = \frac{1}{(k-m\omega^2)^2 + c^2\omega^2} S_0 = \frac{S_0}{(k-m\omega^2)^2 + c^2\omega^2} \quad (93)
\]

Using various values for the elements in the system, the PSD for the system considered looks like:

Another useful quantity to consider is the expected output, in terms of its mean and variance, for a given input. Working within the constraint that the system will oscillate about zero, \(E[x] = 0\), the mean-square value can be directly related to the PSD function as:

\[
E[x^2] = \bar{x}^2 = \int_{-\infty}^{\infty} |H(j\omega)|^2 S_{ff}(\omega) d\omega \quad (94)
\]

For a constant input \(S_0\), as diagrammed in figure 12(b), the mean-square value can be expressed as:

\[
E[x^2] = \bar{x}^2 = S_0 \int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega \quad (95)
\]

After inspecting the above equation, it becomes clear that to obtain the square of the expected value, a solution for \(\int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega\) must be obtained. For cases where \(S_{ff}(\omega) = S_0\) and as such \(S_{ff}(\omega)\) can be pulled out of the integral, these integrals have been solved [Random Vibrations, Spectral & Wavelet Analysis, Newland (1993)]. For example, given \(\int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega\):

\[
\int_{-\infty}^{\infty} \left| \frac{B_0}{A_0 + j\omega A_1} \right|^2 d\omega = \frac{\pi B_0^2}{A_0 A_1} \quad (96)
\]
and
\[
\int_{-\infty}^{\infty} \left| \frac{B_0 + j\omega B_1}{A_0 + j\omega A_1 - \omega^2 A_2} \right|^2 d\omega = \frac{\pi(A_0 B_1^2 + A_2 B_0^2)}{A_0 A_1 A_2} \tag{97}
\]
When combined with equation 95, these integrals allow for the easy calculation of the expected values.

**Example 4.5** For system below, calculate the mean-square response of the system given that the spectrum of the input force \( F(t) \) is a perfect theoretical white noise.

![Figure 14: A spring-dashpot-mass model of a 1-DOF system with external excitation.](image)

**Solution:** Again, as the forcing function \( S_{ff}(\omega) \) is constant across the frequency spectrum \( S_{ff}(\omega) = S_0 \) the mean-square response can be calculated as:
\[
E[x^2] = \bar{x}^2 = S_0 \int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega \tag{98}
\]
Using the already tabulated response:
\[
\int_{-\infty}^{\infty} \left| \frac{B_0 + j\omega B_1}{A_0 + j\omega A_1 - \omega^2 A_2} \right|^2 d\omega = \frac{\pi(A_0 B_1^2 + A_2 B_0^2)}{A_0 A_1 A_2} \tag{99}
\]
and the frequency response function for the system as derived in equation 64:
\[
H(j\omega) = \frac{1}{k - m\omega^2 + c\omega j} \tag{100}
\]
when \( B_0 = 1, B_1 = 0, A_0 = k, A_1 = c, \) and \( A_2 = m \). Therefore, using the tabulated expression we can show that:
\[
E[x^2] = S_0 \frac{\pi m}{kcm} = S_0 \frac{\pi}{kc} \tag{101}
\]
Table of Laplace Transforms for Vibrations

This is a partial list of important Laplace transforms for vibrations that assumes zero initial conditions, \(0 < t\), and \(\zeta < 1\).

<table>
<thead>
<tr>
<th>(f(t))</th>
<th>(\mathcal{L}[f(t)] = F(s))</th>
<th>(f(t))</th>
<th>(\mathcal{L}[f(t)] = F(s))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\delta(t))</td>
<td>1</td>
<td>(\frac{1}{\omega^3}(\omega t - \sin(\omega t)))</td>
<td>(\frac{1}{s^3(s^2 + \omega^2)}) (17)</td>
</tr>
<tr>
<td>(\delta(t - t_0))</td>
<td>(e^{-st_0})</td>
<td>(\frac{1}{2\omega^3}(\sin(\omega t) - \omega t \cos(\omega t))) \ldots</td>
<td>(\frac{1}{(s^2 + \omega^2)^2}) (18)</td>
</tr>
<tr>
<td>1</td>
<td>(\frac{1}{s})</td>
<td>(\frac{s}{s^2 + \omega^2}) (5)</td>
<td>(\frac{s}{(s - \omega)^2 + \omega^2}) (22)</td>
</tr>
<tr>
<td>(e^{at})</td>
<td>(\frac{1}{s - a})</td>
<td>(\frac{t}{2\omega} \sin(\omega t))</td>
<td>(\frac{2\omega s}{(s^2 + \omega^2)^2}) (20)</td>
</tr>
<tr>
<td>(\sin(\omega t))</td>
<td>(\frac{\omega}{s^2 + \omega^2})</td>
<td>(t \sin(\omega t))</td>
<td>(\frac{2\omega s}{(s^2 + \omega^2)^2}) (20)</td>
</tr>
<tr>
<td>(\cos(\omega t))</td>
<td>(\frac{s}{s^2 + \omega^2})</td>
<td>(t \cos(\omega t))</td>
<td>(\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}) (21)</td>
</tr>
<tr>
<td>(\sinh(\omega t))</td>
<td>(\frac{\omega}{s^2 - \omega^2})</td>
<td>(e^{at} \sin(\omega t))</td>
<td>(\frac{\omega}{(s - a)^2 + \omega^2}) (22)</td>
</tr>
<tr>
<td>(\cosh(\omega t))</td>
<td>(\frac{s}{s^2 - \omega^2})</td>
<td>(e^{at} \cos(\omega t))</td>
<td>(\frac{s - a}{(s - a)^2 + \omega^2}) (23)</td>
</tr>
<tr>
<td>(\frac{1}{\omega^2}(1 - \cos(\omega t)))</td>
<td>(\frac{1}{s(s^2 + \omega^2)}) (9)</td>
<td>(e^{at} \sinh(\omega t))</td>
<td>(\frac{s}{(s - a)^2 - \omega^2}) (24)</td>
</tr>
<tr>
<td>(\frac{1}{\omega t}e^{-\zeta \omega t} \sin(\omega_{0t}))</td>
<td>(\frac{1}{s^2 + 2\zeta \omega s + \omega^2}) (10)</td>
<td>(e^{at} \cosh(\omega t))</td>
<td>(\frac{s - a}{(s - a)^2 - \omega^2}) (25)</td>
</tr>
<tr>
<td>(1 - \frac{\omega}{\omega t} e^{-\zeta \omega t} \sin(\omega_{0t} + \phi), \phi = \cos^{-1}(\zeta)\ldots)</td>
<td>(\frac{s^2}{s(s^2 + 2\zeta \omega s + \omega^2)}) (11)</td>
<td>(\frac{1}{\omega_2} \sin(\omega_{2t}) - \frac{1}{\omega_1} \sin(\omega_{1t})\ldots)</td>
<td>(\frac{\omega_1^2 - \omega_2^2}{(s^2 + \omega_1^2)(s^2 + \omega_2^2)}) (26)</td>
</tr>
<tr>
<td>(\frac{t^{n-1}}{(n-1)!}, n = 1, 2, \ldots)</td>
<td>(\frac{1}{s^n}) (12)</td>
<td>(\cos(\omega_{2t}) - \cos(\omega_{1t}))</td>
<td>(\frac{s(\omega_1^2 - \omega_2^2)}{(s^2 + \omega_1^2)(s^2 + \omega_2^2)}) (27)</td>
</tr>
<tr>
<td>(t^n, n = 1, 2, \ldots)</td>
<td>(\frac{n!}{s^{n+1}}) (13)</td>
<td>(e^{at} f(t))</td>
<td>(F(s - a)) (28)</td>
</tr>
<tr>
<td>(t^n e^{\omega t}, n = 1, 2, \ldots)</td>
<td>(\frac{n!}{(s - \omega)^{n+1}}) (14)</td>
<td>(f(t-a)\Phi(t-a))</td>
<td>(e^{-as} F(s)) (29)</td>
</tr>
<tr>
<td>(\frac{1}{\omega}(1 - e^{-\omega t}))</td>
<td>(\frac{1}{s(s + \omega)}) (15)</td>
<td>(\Phi(t-a))</td>
<td>(\frac{e^{-as}}{s}) (30)</td>
</tr>
<tr>
<td>(\frac{1}{\omega^2}(e^{-\omega t} + \omega t - 1))</td>
<td>(\frac{1}{s^2(s + \omega)}) (16)</td>
<td>(f(t)')</td>
<td>(sF(s) - f(0)) (31)</td>
</tr>
</tbody>
</table>
5 Two degree-of-freedom systems

Until now we have only considered and modeled systems that can require one coordinate system to describe their motion. In this chapter we will develop the mathematical tools required to model two-degree-of-freedom system (2-DOF) that require two independent coordinates to describe their motion. As before, the equations that describe the motion of rigid bodies in space are developed from Newton’s second law of motion. However, unlike before, there exists an independent equation for each body in motion. These equations are therefore coupled by the system and are often expressed in matrix notation such that the mass, damping, and stiffness matrices are easily defined.

Review 5.1 The dot product allows us to multiply matrices and is defined as:

\[
\begin{bmatrix}
a & b \\
c & d \\
\end{bmatrix}
\begin{bmatrix}
e \\
f \\
\end{bmatrix} =
\begin{bmatrix}
ae + bf \\
ce + df \\
\end{bmatrix}
\]

Another arrangement of the same principle, in a format more related to vibrations, is:

\[
\begin{bmatrix}
a_1 + a_2 & b \\
c & d \\
\end{bmatrix}
\begin{bmatrix}
e \\
f \\
\end{bmatrix} =
\begin{bmatrix}
(a_1 + a_2)e + bf \\
ce + df \\
\end{bmatrix}
\]

The transpose of a matrix is an operator which flips a matrix over its diagonal. For a matrix \( A \), the transpose \( A^T \) can be written as:

\[
A =
\begin{bmatrix}
a & b \\
c & d \\
e & f \\
\end{bmatrix}
\rightarrow
A^T =
\begin{bmatrix}
a & c & e \\
b & d & f \\
\end{bmatrix}
\]

A matrix is symmetric if \( A = A^T \). Therefore, symmetric matrix must be square and can be written as:

\[
A =
\begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i \\
\end{bmatrix}
= A^T =
\begin{bmatrix}
a & d & g \\
b & e & h \\
c & f & i \\
\end{bmatrix}, \text{ where } b = d, \ c = g, \ f = h
\]

The determinant of a matrix is a value obtained from a square matrix and is often written as \( \det(A) \), \( \det(A) \), or \( |A| \). For a \( 2 \times 2 \) matrix this is defined as:

\[
\det(A) = ad - bc, \text{ when } A =
\begin{bmatrix}
a & b \\
c & d \\
\end{bmatrix}
\]

The inverse of a square matrix is such that \( AA^{-1} = A^{-1}A = I \) where \( I \) is the identity matrix:

\[
I =
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\]

1
and the inverse of a $2 \times 2$ matrix is defined as:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \text{ when } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$  \hspace{1cm} (7)$$

Matrices that do not have an inverse are called a singular matrix.

### 5.1 Modeling Undamped Two Degree of Freedom Systems

Consider the undamped 2-DOF systems presented in figure 1. These system with a single mass capable of moving in two directions. To expand, figure 1(a) reports a mass that can move horizontally or vertically in space. However, this mass does not rotate during its movements. Moreover, figure 1(b) presents a system that rotates about the spring and displaces vertically. These are examples of 2 DOF systems because each system has two independent coordinate systems that express the movement of the mass.

![Figure 1: Examples of single mass 2-DOF systems that: (a) displace in the vertical and horizontal directions; and (b) rotates about the spring and displaces in the vertical direction.](image)

Another example of a 2-DOF system with two masses, each with their own independent coordinate system, is presented in figure 2. The two coordinates that describe the systems movements are $x_1$ and $x_2$.

![Figure 2: 2-DOF system with two masses and two independent confidante systems $x_1$ and $x_2$.](image)
5.1.1 Solution for the Two-Degree-of-Freedom System

Before we derive a model for undamped 2-DOF systems, let us first consider the solution to the system shown in figure 2. The solution consists of two equations, one for each mass. This solution will be derived in section 5.1.2 and is expressed by the coupled equations:

\[ x_1(t) = A_1 \sin(\omega_1 t + \phi_1)u_{11} + A_2 \sin(\omega_2 t + \phi_2)u_{12} \]  
\[ x_2(t) = A_1 \sin(\omega_1 t + \phi_1)u_{21} + A_2 \sin(\omega_2 t + \phi_2)u_{22}, \quad \omega_1 \text{ or } \omega_2 \neq 0 \]  

These two equations can be written as a single equation in matrix form as:

\[ \mathbf{x}(t) = A_1 \sin(\omega_1 t + \phi_1)\mathbf{u}_1 + A_2 \sin(\omega_2 t + \phi_2)\mathbf{u}_2, \quad \omega_1 \text{ or } \omega_2 \neq 0 \]  

Where the bold text denotes vectors. Therefore, the vectors \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) are the mathematical expressions that “couple” or tie the equations together. Expanding these vectors shows:

\[ \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix}, \]  

The four key components of the solution expressed in equation 9 are:

1. \( \omega_1 \) and \( \omega_2 \) are the natural frequencies of the system. They are not the frequencies of the masses. The formulation of the solution states that each mass in general oscillates at two frequencies. Furthermore, suppose that the initial conditions are chosen such that \( A_2 = 0 \). With such an initial condition, each mass oscillates at only one frequency, \( \omega_1 \).

2. \( A_1 \) and \( A_2 \) are the constants of integration and determine the amplitude of the system.

3. \( \phi_1 \) and \( \phi_2 \) represent the phase shift of the system.

4. \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) are the first and second mode shapes of the system and couple the system together.

5.1.2 Deriving the Solution for the Two-Degree-of-Freedom System

To derive this solution for the system under consideration a FBD for figure 2 can be constructed for the forces acting on each mass. First we have to make the assumption that \( x_1 < x_2 \), this allows us to say that \( m_2 \) pulls on \( m_1 \) and results in:

![Figure 3: Free body diagram for the 2-DOM system presented in figure 2.](image-url)
Applying Newton’s second law and summing the forces on each mass in the horizontal direction yields:

\[
\begin{align*}
    m_1 \ddot{x}_1 &= -k_1 x_1 + k_2 (x_2 - x_1) \\
    m_2 \ddot{x}_2 &= -k_2 (x_2 - x_1)
\end{align*}
\] (11)

These equations can be rearranged in terms of \(x_1\) and \(x_2\) as:

\[
\begin{align*}
    m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 &= 0 \\
    m_2 \ddot{x}_2 - k_2 x_1 + k_2 x_2 &= 0
\end{align*}
\] (12)

where these are two coupled second-order differential equations that each require two initial conditions to solve. These initial conditions can be obtained from the displacement and velocity terms as:

\[
\begin{align*}
    x_1(0) &= x_{10} \\
    \dot{x}_1(0) &= \dot{x}_{10} = v_{10} \\
    x_2(0) &= x_{20} \\
    \dot{x}_2(0) &= \dot{x}_{20} = v_{20}
\end{align*}
\] (13)

As before, these initial conditions will be the constants of integration used to solve the two second-order differential equations for the free response of each mass. There is a multitude of ways to solve these two coupled second-order differential equations, however, here we will just consider a matrix notation solution. This matrix notation solution is used as this formulation is readably solved using computers and is expandable to more than 2 DOF.

To initiate the solution, let us first develop the matrix formulation of the two coupled ODEs:

\[
\begin{bmatrix}
    m_1 & 0 \\
    0 & m_2
\end{bmatrix}
\begin{bmatrix}
    \ddot{x}_1 \\
    \ddot{x}_2
\end{bmatrix}
+ \begin{bmatrix}
    k_1 + k_2 & -k_2 \\
    -k_2 & k_2
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix}
= \begin{bmatrix}
    0 \\
    0
\end{bmatrix}
\] (14)

This equation can also be expressed as the vector equation:

\[M \ddot{x} + Kx = 0\] (15)

and is known as the EOM in vector form. In this formulation the mass matrix \(M\) is defined as:

\[
M = \begin{bmatrix}
m_1 & 0 \\
0 & m_2
\end{bmatrix}
\] (16)

while the stiffness matrix \(K\) is:

\[
K = \begin{bmatrix}
k_1 + k_2 & -k_2 \\
-k_2 & k_2
\end{bmatrix}
\] (17)

along with the displacement, velocity, and acceleration matrices:

\[
x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}, \quad \ddot{x} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}
\] (18)
Beyond these equations we can write the initial conditions as:

\[
\mathbf{x}_0 = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}, \quad \dot{\mathbf{x}}_0 = \begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{bmatrix}
\]  

(19)

This simple connection between vibration analysis and matrix analysis allows computers to be used to solve large and complicated vibration problems quickly.

Recall that the 1-DOF version of the equation of motion was solved by assuming a harmonic solution and calculating the values of the constants in the assumed form. The same approach is used here to solve the equation of motion for the two-DOF system. A solution is assumed in the form:

\[
\mathbf{x}(t) = \mathbf{u}e^{j\omega t}
\]  

(20)

where \( \mathbf{u} \) is a vector of constants to be determined and can be written as:

\[
\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\]  

(21)

From before, \( \omega \) is also a constant to be determined. Again, \( j = \sqrt{-1} \). In the same manner as before, \( e^{j\omega t} \) represents harmonic motion as \( e^{j\omega t} = \cos(\omega t) + j\sin(\omega t) \). Taking the derivatives of \( \mathbf{x}(t) = \mathbf{u}e^{j\omega t} \) yields:

\[
\dot{\mathbf{x}}(t) = j\omega \mathbf{u}e^{j\omega t}
\]  

(22)

\[
\ddot{\mathbf{x}}(t) = -\omega^2 \mathbf{u}e^{j\omega t}
\]  

(23)

Substituting this into the EOM in vector form (\( M\ddot{x} + Kx = 0 \)) yields:

\[
-\omega^2 M\mathbf{u}e^{j\omega t} + K\mathbf{u}e^{j\omega t} = 0
\]  

(24)

or

\[
(-\omega^2 M + K)\mathbf{u}e^{j\omega t} = 0
\]  

(25)

As \( e^{j\omega t} \neq 0 \) for any value of \( t \) and not allowing \( \mathbf{u} \) to be zero it can be determined that \( -\omega^2 M + K \) must satisfy the vector equation. Therefore,

\[
(-\omega^2 M + K)\mathbf{u} = 0, \quad \mathbf{u} \neq 0
\]  

(26)

For this homogeneous set of algebraic equations to have a nonzero solution for the vector \( \mathbf{u} \), the inverse of the coefficient matrix \( -\omega^2 M + K \) must not exist. To see that this is the case, suppose that the inverse of \( -\omega^2 M + K \) does exist. Then multiplying both sides of the equation by \( (-\omega^2 M + K)^{-1} \) yields \( \mathbf{u} = 0 \), a trivial solution, as it implies no motion. Hence the solution of equation depends in some way on the matrix inverse.

Applying the condition of singularity to the coefficient matrix of equation \( (-\omega^2 M + K)\mathbf{u} = 0, \ \mathbf{u} \neq 0 \) yields the result that for a nonzero solution of \( \mathbf{u} \) to exist the following must be true:

\[
\det(-\omega^2 M + K) = 0
\]  

(27)
which yields one algebraic equation in one unknown ($\omega$). Substituting the values of the matrices $M$ and $K$ into this expression yields:

$$\det\begin{bmatrix} -\omega^2m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2m_2 + k_2 \end{bmatrix} = 0 \quad (28)$$

Using the definition of the determinant yields that the unknown quantity $\omega^2$ must satisfy:

$$m_1m_2\omega^4 - (m_1k_2 + m_2k_1 + m_2k_2)\omega^2 + k_1k_2 = 0 \quad (29)$$

This expression is called the characteristic equation for the system and is used to determine the constants $\omega_{1,2}$, in the assumed form of the solution given by the assumed solution $x(t) = ue^{j\omega t}$, once the values of the physical parameters $m_1, m_2, k_1,$ and $k_2$ are known. Note that $\omega_{1,2}$ are not in the characteristic equation, therefore, solving for $\omega_{1,2}$ will be done by factoring the equation above to obtain two solutions $\omega_1$ and $\omega_2$. The characteristic equation is in the form of the quadratic formula if you set $x = \omega^2$, as:

$$ax^2 + bx + c = 0 \quad (30)$$

After finding the value of $\omega_{1,2}$ using the characteristic equation, the values in $u$ can be found using equation $(-\omega_1^2M + K)u = 0, \ u \neq 0$ for each value of $\omega^2$. That is, for both $\omega_1$ and $\omega_2$ there is a vector $u$ that satisfies the equation. These solutions can be written as:

$$(-\omega_1^2M + K)u_1 = 0 \quad (31)$$
and

$$(-\omega_2^2M + K)u_2 = 0 \quad (32)$$

These expressions can be solved for the direction of the vectors $u_1$ and $u_2$, but not for the magnitude. To see that this is true, note that if $u_1$, satisfies the equation, so does the vector $au_1$ where $a$ is any nonzero number. Hence the vectors satisfying the above are of arbitrary magnitude.

The values obtained for $u_1$ and $u_2$ can now be combined with the assumed solution:

$$x(t) = ue^{j\omega t} \quad (33)$$
to form a set of solution:

$$x(t) = u_1e^{-j\omega_1 t}, \quad u_1e^{j\omega_1 t}, \quad u_2e^{-j\omega_2 t}, \quad u_2e^{j\omega_2 t} \quad (34)$$

Since the equation to be solved is linear, the solution is the sum of these solutions. This results in:

$$x(t) = (ae^{j\omega_1 t} + be^{-j\omega_1 t})u_1 + (ce^{j\omega_2 t} + de^{-j\omega_2 t})u_2 \quad (35)$$

where $a, b, c,$ and $d$ are the arbitrary constants of integration to be determined by the initial conditions. Applying Euler’s formulas for the sin functions (where $\omega_1$ or $\omega_2 \neq 0$) reorganizes this equation as:

$$x(t) = A_1 \sin(\omega_1 t + \phi_1)u_1 + A_2 \sin(\omega_2 t + \phi_2)u_2, \quad \omega_1 \text{ or } \omega_2 \neq 0 \quad (36)$$
Another way to write this equation is in the form:

\[
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} = [u_1 \quad u_2] \begin{bmatrix} A_1 \sin(\omega_1 t + \phi_1) \\ A_2 \sin(\omega_2 t + \phi_2) \end{bmatrix}, \quad \omega_1 \text{ or } \omega_2 \neq 0
\] (37)

Where the values for \(A_1\) and \(A_2\) can be obtained by setting applying the boundary conditions and taking the derivatives of the equations as done in the 1-DOF problems. The forms of final equations and give physical meaning to the solution. It states that each mass oscillates at the natural frequencies of the system \(\omega_1\) and \(\omega_2\). Furthermore, suppose that the initial conditions are chosen such that \(A_2 = 0\). With such initial conditions each mass oscillates at only one frequency, \(\omega_1\), and the relative positions of the masses at any given instant of time are determined by the elements of the vector \(u_1\). Hence \(u_1\), is called the first mode shape of the system. Similarly, if the initial conditions are chosen such that \(A_1\) is zero, both coordinates oscillate at frequency \(\omega_2\), with relative positions given by the vector \(u_2\), called the second mode shape. The concepts of natural frequencies and mode shapes are extremely important and form one of the major ideas used in vibration studies.

**Example 5.1** Considering the following system:

![Diagram of 2-DOF system with two masses and two independent confidante systems \(x_1\) and \(x_2\).]

Calculate response for the system if \(m_1=9\) kg, \(m_2=1\) kg, \(k_1 = 24\) N/m, and \(k_2 = 3\) N/m with the initial conditions \(x_{10} = 1\) mm, \(v_{10} = 0\) mm/s, \(x_{20} = 0\) mm, and \(v_{20} = 0\) mm/s.

**Solution:** We have already obtained a characteristic equation for this system, given as:

\[m_1 m_2 \omega^4 - (m_1 k_2 + m_2 k_1 + m_2 k_2) \omega^2 + k_1 k_2 = 0\] (38)

Substituting our values into this obtains:

\[9 \cdot 1 \omega^4 - (9 \cdot 3 + 1 \cdot 24 + 1 \cdot 3) \omega^2 + 24 \cdot 3 = 0\] (39)

or

\[\omega^4 - 6\omega^2 + 8 = 0\] (40)

This can then be factored into:

\[(\omega^2 - 2)(\omega^2 - 4) = 0\] (41)

This results in solutions of \(\omega^2_1 = 2\) and \(\omega^2_2 = 4\). Leading to:

\[\omega_1 = \pm \sqrt{2} \text{ rad/sec}, \quad \omega_2 = \pm 2 \text{ rad/sec}\] (42)
Next, we need to obtain solutions for $u_1$ and $u_2$. Having solved for $\omega_1$ and $\omega_2$ we can obtained. First, knowing $u_1 = [u_{11} u_{21}]^T$ and using $\omega_1 = \sqrt{2}$ and the following equation:

$$(-\omega_1^2 M + K)u_1 = 0$$

(43)

yields simplified to

$$
\begin{pmatrix}
-2 & 9 & 0 \\
0 & 1
\end{pmatrix}
+ \begin{pmatrix}
24 & 3 & -3 \\
-3 & 3 & 3
\end{pmatrix}
\begin{pmatrix}
u_{11} \\
u_{21}
\end{pmatrix}
= \begin{pmatrix} 0 \\
0
\end{pmatrix}
$$

(44)

simplified to

$$\begin{pmatrix}
27 & -9 & 2 \\
-3 & 3 & -2
\end{pmatrix}
\begin{pmatrix}
u_{11} \\
u_{21}
\end{pmatrix}
= \begin{pmatrix} 0 \\
0
\end{pmatrix}
$$

(45)

or

$$\begin{pmatrix}
9 & -3 \\
-3 & 1
\end{pmatrix}
\begin{pmatrix}
u_{11} \\
u_{21}
\end{pmatrix}
= \begin{pmatrix} 0 \\
0
\end{pmatrix}
$$

(46)

Taking the dot product of the matrix equation yields:

$$9u_{11} - 3u_{21} = 0, \text{ and } -3u_{11} + u_{21} = 0$$

(47)

Both of these equations yield the same equation, that is:

$$\frac{u_{11}}{u_{21}} = \frac{1}{3}$$

(48)

As mentioned before, only the ratio of the elements is determined here. To show this is true it is easily seen that:

$$u_{11} = u_{21}\frac{1}{3} \rightarrow au_{11} = au_{21}\frac{1}{3}$$

(49)

To obtain a numerical value, we arbitrarily assign a value to one of the elements. Here, let $u_{21} = 1$ so let $u_{11} = 1/3$. Therefore,

$$u_1 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

(50)

The same processes can be used for obtaining $u_1$ using $\omega_2 = 2$, this results in:

$$\begin{pmatrix}
-9 & -3 \\
-3 & -1
\end{pmatrix}
\begin{pmatrix}
u_{12} \\
u_{22}
\end{pmatrix}
= \begin{pmatrix} 0 \\
0
\end{pmatrix}
$$

(51)

Taking the dot product of the matrix equation yields:

$$-9u_{12} - 3u_{22} = 0, \text{ and } -3u_{12} - u_{22} = 0$$

(52)

Both of these equations yield the same equation, that is:

$$\frac{u_{12}}{u_{22}} = -\frac{1}{3}$$

(53)
Again, assuming \( u_{22} = 1 \) this can be rearranged into \( u_2 \) as:

\[
\begin{bmatrix} u_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\]

(54)

Where \( u_1 \) and \( u_2 \) represent only the directions and shape of the mode shapes and not the magnitude of the mode shapes. Now that we have the mode shapes, we can solve for the initial conditions \( A_1 \) and \( A_2 \). To do this, let us use the following formulation of the solution:

\[
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \begin{bmatrix} A_1 \sin(\omega_1 t + \phi_1) \\ A_2 \sin(\omega_2 t + \phi_2) \end{bmatrix}, \quad \omega_1 \text{ or } \omega_2 \neq 0
\]

(55)

Adding our values for the problem at \( t = 0 \) this becomes:

\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} A_1 \sin(\sqrt{2}t + \phi_1) \\ A_2 \sin(2t + \phi_2) \end{bmatrix}
\]

(56)

and after applying the dot product:

\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}A_1 \sin(\phi_1) - \frac{1}{3}A_2 \sin(\phi_2) \\ A_1 \sin(\phi_1) + A_2 \sin(\phi_2) \end{bmatrix}
\]

(57)

Next we can differentiate the equation for \( x(t) \) to obtain the velocity solution. Adding our values for the problem at \( t = 0 \) obtains:

\[
\begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{bmatrix} = \begin{bmatrix} v_{10} \\ v_{20} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{3}A_1 \cos(\phi_1) - \frac{2}{3}A_2 \cos(\phi_2) \\ \sqrt{2}A_1 \cos(\phi_1) + 2A_2 \cos(\phi_2) \end{bmatrix}
\]

(58)

Now that we have 4 equations for 4 unknowns we can use these equations to solve for \( A_1, A_2, \phi_1, \) and \( \phi_2 \). The 4 equations are:

\[
3 = A_1 \sin(\phi_1) - A_2 \sin(\phi_2)
\]

(59)

\[
0 = A_1 \sin(\phi_1) + A_2 \sin(\phi_2)
\]

(60)

\[
0 = \sqrt{2}A_1 \cos(\phi_1) - 2A_2 \cos(\phi_2)
\]

(61)

\[
0 = \sqrt{2}A_1 \cos(\phi_1) + 2A_2 \cos(\phi_2)
\]

(62)

Setting these last two equations equal to each other yields:

\[
0 = \sqrt{2}A_1 \cos(\phi_1) + 2A_2 \cos(\phi_2) = \sqrt{2}A_1 \cos(\phi_1) - 2A_2 \cos(\phi_2)
\]

(63)

or:

\[
0 = -4A_2 \cos(\phi_2)
\]

(64)
For this equation to be true, $\phi_2 = \frac{\pi}{2}$. Therefore, applying this to $0 = \sqrt{2}A_1 \cos(\phi_1) + 2A_2 \cos(\phi_2)$ results in:

$$0 = \sqrt{2}A_1 \cos(\phi_1) \quad (65)$$

where again, for this equation to be true, $\phi_1 = \frac{\pi}{2}$. Now the first two equations become:

$$3 = A_1 - A_2 \quad (66)$$

$$0 = A_1 + A_2 \quad (67)$$

Where this shows us that $A_1 = \frac{3}{2}$ mm and $A_2 = -\frac{3}{2}$. Therefore, now that we have the initial conditions we can find a solution for the temporal response of each mass. Using the equations from before:

$$x_1(t) = A_1 \sin(\omega_1 t + \phi_1)u_{11} + A_2 \sin(\omega_2 t + \phi_2)u_{12} \quad (68)$$

$$x_2(t) = A_1 \sin(\omega_1 t + \phi_1)u_{21} + A_2 \sin(\omega_2 t + \phi_2)u_{22} \quad (69)$$

And applying our obtained values

$$x_1(t) = \frac{3}{2} \sin(\sqrt{2}t + \frac{\pi}{2}) + \left(-\frac{3}{2}\right) \sin(2t + \frac{\pi}{3}) \left(-\frac{1}{3}\right) \quad (70)$$

$$x_2(t) = \frac{3}{2} \sin(\sqrt{2}t + \frac{\pi}{2}) + \left(-\frac{3}{2}\right) \sin(2t + \frac{\pi}{2}) \quad (71)$$

results in:

$$x_1(t) = \frac{1}{2} \left( \sin(\sqrt{2}t + \frac{\pi}{2}) + \sin(2t + \frac{\pi}{2}) \right) \quad (72)$$

$$x_2(t) = \frac{3}{2} \left( \sin(\sqrt{2}t + \frac{\pi}{2}) - \sin(2t + \frac{\pi}{2}) \right) \quad (73)$$

These results can be plotted as:
5.1.3 Mode Shapes

Studying and characterizing the natural frequencies of a system allows for the detailed investigation of the system response. Modern vibration analysis relies heavily on the concepts of mode shapes for various engineering tasks. Mode shapes can be better understood through a graphical representation. To do this, consider the 2-DOF system presented in figure 6(a). Assuming that $x_1 < x_2$ the FBD for the system is expressed in figure 6(b).

![Figure 6: (a) 2-DOF system with two masses arranged in a vertical configuration; and (b) FBD of system.](image)
For simplicity, all masses and spring stiffness are considered equal and that $m = 1$ and $k = 1$. From the previous investigations in this text we know that the forces caused by gravity will cancel out. Therefore, the EOM for the system can be written as:

$$m_1 \ddot{x}_1 = -kx_1 + k(x_2 - x_1) \quad (74)$$
$$m_2 \ddot{x}_2 = -k(x_2 - x_1) - kx_2$$

These equations can be written in matrix notation as:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (75)$$

Substituting the values of the matrices $M$ and $K$ into this expression $\det(-\omega^2 M + K) = 0$ yields:

$$\det \begin{bmatrix} -\omega^2 m + 2k & -k \\ -k & -\omega^2 m + 2k \end{bmatrix} = 0 \quad (76)$$

The determinant yields that the unknown quantity, $\omega^2$, must satisfy:

$$m^2 \omega^4 - 4km\omega^2 + 3k = 0 \quad (77)$$

Therefore,

$$\omega_1 \pm \sqrt{\frac{k}{m}} = 1 \text{ rad/sec}, \quad \omega_2 \pm \sqrt{\frac{3k}{m}} = \sqrt{3} \text{ rad/sec} \quad (78)$$

Now, we need to obtain solutions for $u_1$ and $u_2$. Knowing $(-\omega_1^2 M + K)u_1 = 0$ yields:

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (79)$$

Taking the dot product of the matrix equation yields:

$$u_{11} - u_{21} = 0, \quad \text{and} \quad -u_{11} + u_{21} = 0 \quad (80)$$

Setting $u_{11} = 1$ results in $u_{21} = 1$. The same processes can be performed for $u_2$ to show that if we set $u_{12} = 1$, $u_{22} = -1$. Therefore, the mode shapes can be expresses as:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (81)$$

The displacement of the masses as a function of time and the general mode shape plots are graphically represented in figure 7. In the 2-DOF system considered here, the second mode shape has a spot at the center of the middle spring spring that does not move (i.e. has zero displacement). This point is called a node. Nodes correspond to points in the mode shape where the displacement is always zero. Furthermore, the displacement of the node points remain zero at all times, as diagrammed in the top-right of figure 7.
5.2 Eigenvalue-based Solution for Natural Frequencies and Mode Shapes

The process of calculating the mode shapes presented in sections 5.1.2 and 5.1.3 is long and tedious. Therefore, methods that can be easily deployed on computers are of great interest to the practitioner. An eigenvalue-based solution that takes advantage of the symmetry in the $M$ and $K$ matrices and can be easily implemented on a computer is discussed in this section.

Review 5.2 In linear algebra, eigenvalues and eigenvectors are concepts that appear prominently in the analysis of linear transformations. By definition, if $v$ is a vector (in vector space $V$ over a field $F$) and $T$ is a linear transformation into itself, then $v$ is an eigenvector of $T$ if $Tv$ is a scalar multiple of $v$:

$$Tv = \lambda v$$  \hspace{1cm} (82)

where $\lambda$ is a scalar in the field $F$, known as the eigenvalue associated with the eigenvector $v$.

A matrix is positive definite if the scalar $x^T Ax$ is positive for any non-zero vector $x$ comprised of real numbers:

$$x^T Ax > 0$$  \hspace{1cm} (83)

The Cholesky decomposition of a real positive-definite matrix $A$ is a decomposition of the form:

$$A = LL^T$$  \hspace{1cm} (84)

where $L$ is the lower triangular matrix of $A$. 

Figure 7: Modes of vibration for the system shown in figure 6 showing the: (a) first mode; and (b) second mode.
5.2.1 Cholesky Decomposition

The vast majority of mass and stiffness matrices are symmetric and positive definite due to the physical meaning of these matrices. Therefore, $M$ can be factored into two terms using the Cholesky decomposition:

$$M = LL^T$$  \hspace{1cm} (85)

For diagonal mass matrices (all the mass values lie along the diagonal of the matrix) the Cholesky decomposition ($L$) is defined as:

$$L = M^{1/2} = \begin{bmatrix} \sqrt{m_1} & 0 \\ 0 & \sqrt{m_2} \end{bmatrix}$$  \hspace{1cm} (86)

This expression factors into:

$$M = M^{1/2}M^{1/2}$$  \hspace{1cm} (87)

Moreover, the inverse of the diagonal matrix ($M^{1/2}$) is denoted as $M^{-1/2}$ and defined as:

$$L^{-1} = M^{-1/2} = \begin{bmatrix} \frac{1}{\sqrt{m_1}} & 0 \\ 0 & \frac{1}{\sqrt{m_2}} \end{bmatrix}$$  \hspace{1cm} (88)

5.2.2 Deriving the Eigenvalue-based Solution

To derive an eigenvalue-based solution for calculating the natural frequencies and mode shapes let us consider the previously derived EOM for a undamped 2-DOF system:

$$M\ddot{x} + Kx = 0$$  \hspace{1cm} (89)

This expression can be transformed into a symmetric eigenvalue problem, allowing us to leverage the strengths of symmetric eigenvalue mathematics and computer solvers. To perform this transform, we set $x = M^{-1/2}q$ and multiply the equation by $M^{-1/2}$ such that the EOM becomes:

$$M^{-1/2}MM^{-1/2}\ddot{q} + M^{-1/2}KM^{-1/2}q = 0$$  \hspace{1cm} (90)

As $M^{-1/2}MM^{-1/2}$ is equal to the identity matrix $I$ and defining $M^{-1/2}KM^{-1/2}$ as the mass normalized stiffness $\tilde{K}$ yields the simplified expression:

$$I\ddot{q} + \tilde{K}q = 0$$  \hspace{1cm} (91)

where $\tilde{K} = M^{-1/2}KM^{-1/2}$ is analogous to $\sqrt{k/m}$ from the 1-DOF system.

As before, a solution is found by assuming a solution, taking the derivatives of the solution and substituting it into the EOM. Following these steps and assuming a solution of:

$$q = ve^{j\omega t}$$  \hspace{1cm} (92)

results in an EOM in the form:

$$-v\omega^2e^{j\omega t} + \tilde{K}ve^{j\omega t} = 0$$  \hspace{1cm} (93)

diving out the nonzero scaler $e^{j\omega t}$ and rearranging the above expression results in:

$$\tilde{K}v = \omega^2v$$  \hspace{1cm} (94)
Knowing that \( \mathbf{v} \neq 0 \), as a vector with zeros would mean no motion is present in the system, this equation can be expressed in a typical eigenvalue formulation:

\[
\tilde{K} \mathbf{v} = \lambda \mathbf{v}
\]  

(95)

where \( \lambda = \omega^2 \). Or more importantly, \( \omega = \sqrt{\lambda} \). As \( \tilde{K} \) is symmetric, this is a symmetric eigenvalue problem. Moreover, the vector \( \lambda \) represent the eigenvalues of the system. Given that we set \( \mathbf{x} = M^{-1/2} \mathbf{q} \), the eigenvectors are not a direct representation of the mode shapes. To develop a link between the eigenvectors and mode shapes we first need to normalize the lengths of the eigenvectors obtained by solving equation 95 to that of a unit vector. The norm of a unit vector is defined as:

\[
1 = ||\mathbf{v}|| = \sqrt{\mathbf{v}^T \mathbf{v}} = \sqrt{\sum_{i=1}^{n} (v_i^2)}
\]  

(96)

where a scalar is used in conjunction with vector such that \( \alpha \mathbf{u} = 1 \) computed the normalized unit vector directions as outlined in example 5.2. In general, a nonzero vector of any length can be normalized by calculating:

\[
\frac{1}{\sqrt{\mathbf{v}^T \mathbf{v}}} \mathbf{v}
\]  

(97)

We can relate the eigenvectors to the modes shapes by a factor of the mass matrix:

\[
\mathbf{u}_1 = M^{-1/2} \mathbf{v}_1
\]  

(98)

The important thing to remember is that the natural frequencies are the square root of the eigenvalues and the mode shapes are related to the eigenvectors through the mass matrix.

**Example 5.2** Normalize the vector \( \mathbf{v}_1 = [1/3 \ 1]^T \)

**Solution:** To normalize the vector \( \mathbf{v}_1 \), a scalar (\( \alpha \)) is calculated to make \( \alpha \mathbf{v} = 1 \). Therefore, following the definition of an orthogonal vector:

\[
(\alpha \mathbf{v}_1)^T (\alpha \mathbf{v}_1) = 1
\]  

(99)

or:

\[
\alpha [1/3 \ 1] \alpha \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} = \alpha^2 (1/9 + 1) = 1
\]  

(100)

Therefore, \( \alpha = 3/\sqrt{10} \). Resulting in a normalized unit vector of \( \alpha \mathbf{v}_1 = [1/\sqrt{10} \ 3/\sqrt{10}]^T \)

**Example 5.3** Consider the system presented in example 5.1 and repeated below where \( m_1 = 9 \) kg, \( m_2 = 1 \) kg, \( k_1 = 24 \) N/m, and \( k_2 = 3 \) N/m with the initial conditions \( x_{10} = 1 \) mm, \( v_{10} = 0 \) mm/s, \( x_{20} = 0 \) mm, and \( v_{20} = 0 \) mm/s. Calculate the natural frequencies and the mode shapes using the eigenvalue solution.
Figure 8: 2-DOF system with two masses and two independent confidante systems $x_1$ and $x_2$.

**Solution:** Writing the mass and stiffness matrix of the system as:

$$M = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$$  \hspace{1cm} (101)

and

$$K = \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}$$  \hspace{1cm} (102)

we can compute $\tilde{K}$ using the following expression:

$$\tilde{K} = M^{-1/2}KM^{-1/2}$$  \hspace{1cm} (103)

where $KM^{-1/2}$ is computed first to maintain symmetry. This results in:

$$KM^{-1/2} = \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -3 \\ -1 & 3 \end{bmatrix}$$  \hspace{1cm} (104)

and:

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 9 & -3 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 3/3 & -1 \\ -1 & 3 \end{bmatrix}$$  \hspace{1cm} (105)

Now a solution must be obtained for the eigenvalue problem:

$$\tilde{K}v = \lambda v$$  \hspace{1cm} (106)

While this can be obtained using computers for such a simple case it is more appropriate to solve this expression by hand. Therefore, the above expression can be rewritten as:

$$(\tilde{K} - \lambda I)v = 0$$  \hspace{1cm} (107)

as $v \neq 0$ the matrix must be singular, resulting in the determinant of the matrix equaling zero. Or:

$$\det \begin{bmatrix} 3 - \lambda & -1 \\ -1 & 3 - \lambda \end{bmatrix} = 0$$  \hspace{1cm} (108)

This can be expanded to the characteristic equation:

$$\lambda^2 - 6\lambda + 8 = 0$$  \hspace{1cm} (109)
with the roots (eigenvalues):

\[ \lambda_1 = 2 \text{ and } \lambda_2 = 4 \]  

(110)

Therefore, \( \omega_1 = \sqrt{2} \) and \( \omega_2 = 2 \). These are the same values computed in example 5.1. The eigenvectors for \( \lambda_1 \) are computed as:

\[
(\tilde{K} - \lambda_1 I)v = 0
\]

(111)

or:

\[
\begin{bmatrix}
3 - 2 & -1 \\
-1 & 3 - 2
\end{bmatrix}
\begin{bmatrix}
v_{11} \\
v_{21}
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

(112)

This results in two dependent scalar equations:

\[ v_{11} - v_{21} = 0 \text{ and } -v_{11} + v_{21} = 0 \]  

(113)

That show us that \( v_{11} = v_{21} \) or \( v_1 = [1 \ 1]^T \). Therefore, using \( (\alpha v_1)^T (\alpha v_1) = 1 \) we obtain:

\[ \alpha [1 \ 1] \alpha [1] = \alpha^2 (2) = 1 \]  

(114)

or \( \alpha = 1/\sqrt{2} \). This allows us to normalize the vector knowing \( \alpha v_1 = 1 \), resulting in a normalized vector of:

\[
\alpha v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

(115)

A similar processes is followed for \( \lambda_2 = 4 \) that leads to the normalized vector

\[
\alpha v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\]

(116)

Lastly, the normalized eigenvectors can be converted to mode shapes using \( u = M^{-1/2}v \). Resulting in:

\[
u_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \end{bmatrix}
\]

(117)

and:

\[
u_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 1 \end{bmatrix}
\]

(118)

Therefore, the first mode shape is \([1 \ 1]\) while the second mode shape is \([\frac{1}{\sqrt{2}} \ -\frac{1}{\sqrt{2}}]\). Again, this is using our prior definition of mode shapes to set \([1 \ 1]\) as the first mode shape. Note that these are the same mode shape vectors as computed in example 5.1.
6 Vibration Control

Throughout this text, we have studied various aspects related to analyzing and modeling vibrating systems. Therefore, it becomes prudent to look at methods for reducing or eliminating unwanted vibrations. However, before vibrations in a system can be effectively reduced they must be better understood in terms of their effects on the system under study. For this reason, this chapter first introduces the vibration Nomograph, this is then followed by vibration isolation, absorption, and active suppression.

6.1 Vibration Nomograph

There exist various methods and standards for measuring and describing acceptable levels of vibrations in systems, these include ISO/AWI 2631 for the evaluation of human exposure to whole-body vibrations and ISO 4866 for the measurement and effects of vibrations on structures. A common way to present the acceptable limit of vibration is in a vibration nomograph. A vibration nomograph is a simplified way to express the acceptable limits on a system while considering the displacement, velocity, acceleration, and frequency of a system. A typical nomograph with various limits is presented in figure 1.

A vibration nomograph is a logarithmic plot that allows us to easily express the relationships between displacement, velocity, acceleration, and frequency of a system. The vibration nomograph presented in figure 1 considers an undamped 1-DOF system with constant amplitude \( A \) experiencing harmonic motion that can be modeled as:

\[
x(t) = A \sin(\omega t)
\]

Therefore, the velocity and acceleration terms can be found by taking the derivatives of the displacement expression to yield:

\[
\dot{x}(t) = A\omega \cos(\omega t)
\]

and:

\[
\ddot{x}(t) = -A\omega^2 \sin(\omega t)
\]

These equations are converted from a circular frequency in rad/sec to a linear frequency \( f \) in Hz, such that \( \omega = 2\pi f \). Therefore, equations 1-3 become:

\[
x(t) = A \sin(\omega t)
\]

\[
v(t) = \dot{x}(t) = 2\pi f A \cos(\omega t)
\]

\[
a(t) = \ddot{x}(t) = -4\pi^2 f^2 A \sin(\omega t)
\]

Thereafter, the maximum values for velocity \( v_{\text{max}} \) and acceleration \( a_{\text{max}} \) are related to amplitude through:

\[
v_{\text{max}} = 2\pi f A
\]

\[
a_{\text{max}} = -4\pi^2 f^2 A = -2\pi f v_{\text{max}}
\]

by taking the natural log of both side of equation 7 we obtain:

\[
\ln v_{\text{max}} = \ln 2\pi f + \ln A
\]
doing the same for equation 8 and rearranging in terms of \( v_{\text{max}} \) leads to:

\[
\ln v_{\text{max}} = -\ln a_{\text{max}} - \ln 2\pi f
\]  

(10)

It can be seen that both of these expressions are linear. Moreover, for a constant amplitude of displacement, \( A \), equation 9 tells us that \( \ln v_{\text{max}} \) varies with \( \ln(2\pi f) \). As the \( x \)-axis in a nomograph is frequency, measured in Hz, \( \ln(2\pi f) \) is a straight line with a positive slope of 1. Therefore, a line on the nomograph that represents a constant displacement is at a 45° angle from the \( x \)-axis. Similarly, equation 10 shows that for a constant value of acceleration (\( a_{\text{max}} \)) the \( \ln v_{\text{max}} \) is represented by a straight line that varies with \( -\ln(2\pi f) \), therefore, \( \ln v_{\text{max}} \) is a straight line with the slope of -1. This is also represented by a line of constant acceleration set at a -45° angle from the \( x \)-axis. These equations are expressed in the vibration nomograph plot of figure 1 where each point on the plot represents a specific sinusoidal (harmonic) vibration for a 1-DOF system.
6.2 Vibration Isolation

6.3 Vibration Absorption

6.4 Active Vibration Suppression

Active vibration control add energy to the system in order to mitigate the vibrations in the system. As depicted in figure 2(a), an active vibration control system requires a sensor to acquire data from the system, control hardware and algorithms to processes this data, and an actuator to exert a physical control on the system. These system together are called a feedback loop, as a movement in the mass results in a control force \( f_u \) being exerted on the system. This control force is
Adding the control force to the EOM for the 1-DOF system presented in figure 2 results in:

\[ m \ddot{x} + c \dot{x} + kx = F(t) = f + f_u \]  (11)

A common method for providing control for vibration suppression is called position derivative control or PD-control. PD-control measures the position and velocity of the mass and uses these to compute the control force needed to mitigate the vibration to an acceptable level. A simple way to code a PD-controller is to provide a control force proportional to the displacement velocity (derivative of displacement) of the mass such that:

\[ f_u = -g_1x - g_2\dot{x} \]  (12)

where \( g_1 \) and \( g_2 \) are the proportional gains of the systems. The control gains can be constants determined by the designer or variables updated through time by an algorithm. Here we will consider the gains to be constant, therefore, the EOM for the closed-loop system in figure 2 becomes:

\[ m \ddot{x} + (c + g_2)\dot{x} + (k + g_1)x = F(t) = f \]  (13)

This formulation lets \( g_1 \) act as additional damping while \( g_2 \) acts as additional damping. This closed-loop EOM can be used to solve for the effective natural frequency of the system, given by:

\[ \omega_n = \sqrt{\frac{k + g_p}{m}} \]  (14)

and the effective damping ratio of the system

\[ \zeta = \frac{c + g_2}{2\sqrt{m(k + g_1)}} \]  (15)
References Cited
